Syllabus

Paper IV - Section I

Relativity (50 Marks)

1. Galilean Transformations, Newtonian Relativity, Electromagnetism and Newtonian Relativity, Michelson-Morley Experiment, Lorentz-Fitzgerald Contraction, Ether-drag hypothesis, Postulates of Special Theory of Relativity.
2. Relativistic Kinematics :

   Relativity of Simultaneity, Derivation of Lorentz Transformation Equations, Some consequences of the Lorentz Transformation Equations, Main features of Lorentz Transformation Equations, Relativistic addition of velocities, Aberration and Doppler Effects in Relativity.
3. Relativistic Mechanics :

   Mechanics and Relativity, Redefining momentum, Relativistic Momentum, Mass in Relativity, Relativistic Force Law and Dynamics of a single particle, Equivalence of Mass and Energy, Transformation properties of Momentum, Energy, Mass and Force.
4. Relativity and Electromagnetism :

   Interdependence of Electric and Magnetic Fields, Transformation Formulae for E and B, Field of uniformly moving point charge, Forces and Fields near a current carrying wire, Forces between moving charges.
5. Geometric Representation of Space-Time.
6. The Twin Paradox.
7. Principle of Equivalence and General Relativity.

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Introduction

In 1905, Albert Einstein put forward the first version of the theory of Relativity. According to this theory, the three fundamental quantities of Physics, namely, space (L), time (T) and mass (M) are not absolute but relative i.e. their values are different for different observers whose frames of reference are in different states of motion. In other words, the value of each quantity varies from observer to observer in different states of motion. Due to this, all other quantities of Physics such as velocity, acceleration, momentum, work, power, energy etc. which depend on (M), (L) and (T) also become relative. Hence this theory is called the theory of Relativity.

The theory which was put forward in 1905 deals with a special type of the frames of reference called inertial frames of reference (i.e. the frames in uniform translational motion). Hence the theory was named as the "Special theory of Relativity".

In 1916, Einstein formulated a new theory which deals with all types of frames of reference, inertial as well as non-inertial (i.e. the frames in non-uniform motion or rotational motion). This theory was named as the "General theory of Relativity".

Special and General theories of Relativity are probably some of the most beautiful and the most perfect of all existing theories of Physics. They are the most logically complete and satisfying theories in existence. Their foundations are so real that no physicist has found any reason to reject any one of them till the present day.

As we shall see in this book, the theories of Relativity created a great revolution in the fundamental concepts of Physics and Human thinking. They produced complete transformations in classical ideas of space, time and mass. They provided a thorough refinement of Classical Newtonian Mechanics for high velocities, comparable to the velocity of light. The most beautiful feature of these theories is that they reduce to the theories of Classical Newtonian Mechanics in the limits of low velocities.

The theories of Relativity have stood immortal and unshaken i the winds of time. All experimental investigations carried out to tes their validity have contributed to man's faith in Relativity.

The theories of Relativity are the greatest achievements o human intellect and genius of Albert Einstein who is credited as a sole originator and creator of these theories. They are fruits of the tremendous power of human thinking and devotion. Einstein's name has become almost synonymous with Relativity.

We shall present the classical, pre-relativistic ideas and considerations in the first chapter under the title "Newtonian Relativity". These ideas will be revised and the new ideas of the Special theory of Relativity will be formulated in the chapter 2. Relativistic Kinematics will be developed in this chapter. The chapter 3 will be devoted to the formulation of Relativistic Mechanics. Relativistic Electromagnetism will be formulated in chapter 4. Space-time diagrams will be presented in chapter 5. A twin paradox or clock paradox will be investigated in chapter 6. In the last chapter (i.e. chapter 7), we shall study the elementary principles and conclusions of the General theory of Relativity.

* * *

1

Newtonian Relativity

In this chapter we shall study the concepts and considerations of classical Relativity, which were prevalent before 1905 i.e.before Albert Einstein put forward the Special theory of Relativity. The classical Relativity was based on Newton's ideas of space, time and mass. Hence we call it Newtonian Relativity.

Using Newtonian Relativity we shall arrive at some theoretical conclusions in Classical Mechanics and Electrodynamics. We shall describe some epoch - making experiments which were devised to verify these theoretical conclusions. We shall observe that the experimental results do not agree with the theoretical conclusions. Here you will realize the necessity of revising the basic ideas of Newtonian Relativity. Only after convering this background you will be able to understand and appreciate the new revolutionary concepts and formulations introduced by the Special theory of Relativity to be presented in the next chapter.

1.1 FRAME OF REFERENCE :

In mechanics we study different types of motions. These motions are always described relative to some reference body. Moreover the description of the motion also depends upon the reference body selected by us. To understand this let us consider the following example.

An observer drops a stone from a railway carriage which is travelling with uniform velocity. He observes from the carriage that the stone falls onto the ground in vertically downward direction (neglecting the effect of air resistance). The same experiment is observed by another observer standing on the ground. He finds that the stone moves to the ground along a parabolic curve. Thus, if the railway carriage is taken as a reference body the motion is linear. If the ground is taken as a reference body the motion is parabolic. You can see clearly how the description of motion depends upon the reference body relative to which the motion is described.

To describe the motion mathematically we have to locate the positions of the particle in space at different instants of time. If we use Cartesian co-ordinate system the position of the particle can be described by the co-ordinates (x,y,z). To locate the instants of time a stop-watch (which can be started suitably) can be provided in the cartesian co-ordinate system. We can attach this system along with the stop-watch to the reference body. This becomes one frame of reference. Such frame of reference can be fixed in a suitable manner to the railway carriage as well as to the ground.

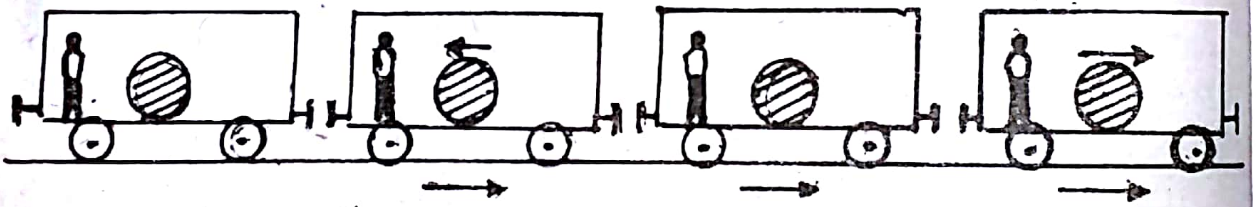
The frames of reference can be classified into two catagories (i) Inertial frames of reference (ii) Non-inertial frames of reference.

1.2 INERTIAL FRAME OF REFERENCE :

Definition : A frame of reference in which Newton's law of inertia (i.e. Newton's first law of motion) holds good is called an inertial frame of reference.

According to Newton's law of inertia, a body continues in its state or rest or of uniform motion along a straight line unless and until an external force acts on it. A frame of reference in which this actually happens is called an inertial frame of reference. In order to understand this, we shall consider the following example.

Example:-



a) Frame at rest b) Frame in Accelerated Motion c) Frame in Uniform Motion d) Frame in Retarded Motion

Fig. 1.1 : A railway carriage at rest and in uniform/non-uniform motion

A sphere is lying on the floor of a railway carriage (Fig 1.1). The carriage is completely closed and the observer standing in the carriage has no knowledge of what is happening outside the carriage. The observer does not apply any external force on the sphere.

a) When the carriage is stationary the observer inside the carriage finds that the sphere remains stationary relative to the carriage unless and until he applies some force on it. i.e. the sphere obeys Newton's law of inertia relative to the carriage. He concludes that the carriage is an inertial frame.

b) The carriage is pulled abruptly towards right so that it moves in the same direction with an acceleration. The observer of the carriage finds that suddenly the sphere moves towards him [Fig 1.1(b)] with an acceleration, by itself, automatically, without application of any external force. The observer concludes that, now, the sphere does not obey Newton's law of inertia relative to the carriage and the carriage is not an inertial frame.

c) After some time, the carriage attains high velocity and moves with the same uniform velocity along a straight line. The observer inside the carriage observes that again the sphere obeys Newton's law of inertia i.e. unless and until he applies some force, the sphere remains at rest relative to the carriage. Hence he concludes that once again the carriage has become an inertial frame.

d) Suddenly the carriage is retarded. Its velocity goes on decreasing. The observer of the carriage finds that suddenly the sphere starts moving away from him [Fig.1.1(d)] automatically, without application of any external force. The observer concludes that again the sphere does not obey Newton's law of inertia and the carriage is no longer an inertial frame.

Thus when the railway carriage is stationary or in uniform translational motion it is an inertial frame of reference. When it is accelerated or retarded or when it is changing the direction of motion it no longer behaves as an inertial frame of reference.

You must have experienced that if you toss a ball in vertically upward direction (Fig. 1.2) in a railway carriage having uniform translational motion it drops back into your hands. Its behaviour is the same as that in a stationary carriage. This is due to the fact that both the frames of reference are inertial frames of reference in which Newton's law of inertia is obeyed identically. In this respect both the frames are equivalent.

Fig. 1.2 : A ball tossed in vertically upward direction drops back into your hands when the train is in uniform translational motion.

If the railway carriage is accelerated or retarded suddenly or it is moving along a sharply curved track the ball tossed in vertically upward direction would not drop into your hands because now the frame of reference is not inertial frame and Newton's law of inertia is not obeyed relative to it.

Corollary : If two frames of reference are moving with an uniform velocity relative to each other and if one of them is known to be an inertial frame then the other frame becomes an inertial frame automatically. This is due to the fact that the given inertial frame must be in uniform translational motion. Since the relative velocity between the two frames is uniform the other frame also is in uniform translational motion and hence it is an inertial frame.

1.3 NON-INERTIAL FRAME OF REFERENCE :

Definition : A frame of reference which is not inertial is called a non-inertial frame of reference.

In a non-inertial frame of reference, Newton's law of inertia does not hold good. i.e. a body may change its state of rest or of uniform motion even without application of any external force. This will be clear from the following examples.

Examples :

i) A railway carriage which is accelerated or retarded or which is changing the direction of motion is a non-inertial frame of reference (Fig 1.1).

ii) A rotating merry-go-round is a non-inertial frame of reference. To test this, let us put a sphere on the floor of the merry-go-round and set the merry-go-round into rotation. The sphere starts moving away from the axis of rotation relative to the merry-go-round automatically, without application of any external force. Thus in this frame the law of inertia is not obeyed and it is a non-inertial frame.

iii) Earth rotates about its own axis. It is orbiting around the sun. Hence it is a non-inertial frame.

Remarks :

1. Strictly speaking there is no perfect inertial frame of reference in the universe. Ideal inertial frame exists only in theory. Newton had assumed that a frame of reference fixed with respect to the stars is an inertial frame of reference. But now it is known that even the stars may be in rotational or some other non-uniform motion.

2. Newton's laws of motion are true only in inertial frames of reference and as we have seen, there is no truely perfect inertial frame of reference in existence. You can visualize the limited scope of Newton's laws of Motion and Classical Mechanics which is based on these laws. Even Newton was very much disturbed by the fact that his laws of motion have such restricted validity and limited applicability.

3. For practical purposes we can neglect the effects produced by the rotational and orbital motion of the earth. Then a frame of reference attached to the earth can be taken as an inertial frame of reference. Due to this, the other frames in uniform translational motion relative to the earth can be considered as inertial frames.

Note : In special theory of Relativity we consider only the inertial frames of reference. Hence the theory has been named as "Special" theory of Relativity.

In General theory of Relativity we consider all frames of reference, inertial as well as non-inertial. Hence the theory is called "General" theory of Relativity.

1.4 EVENTS AND CO-ORDINATE TRANSFORMATIONS :

An event is something occuring at a point in space at an instant of time. To describe an event relative to a frame of reference four co-ordinates (x,y,z,t) are required. They will give information about - where the event has occured and when it has occured.

The description of an event may vary from one frame of reference to the other. For example in a frame S an event is described as (x,y,z,t). Relative to the frame S′ the same event is described as (x',y',z',t'). If we can establish equations connecting these co-ordinates then the description of an event observed in one frame can be transformed into another frame. Such equations are called co-ordinate transformation equations. We shall now obtain such transformation equations of Newtonian Relativity. They are named as Galilean transformation equations (or classical transformation equations).

1.5 GALILEAN TRANSFORMATIONS :

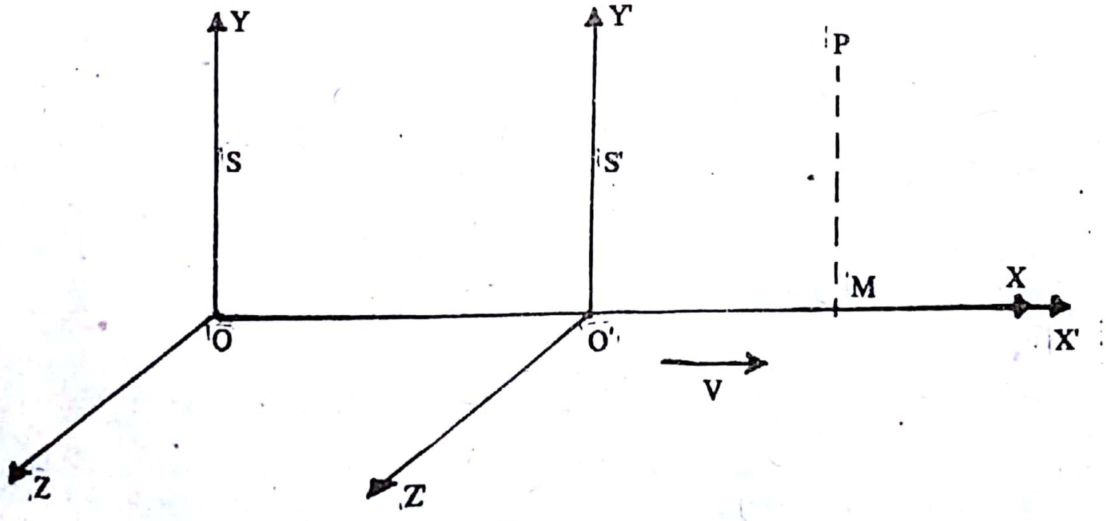


Fig 1.3 : Two Inertial frames of reference in relative motion

Let us consider two inertial frames of reference S and S′. The frame S′ is moving with uniform linear velocity V (use capital letter) relative to S along X-axis so that X-axis and X'-axis coincide with each other.

When the origin O′ coincides with O momentarily the stop-watches are put on in the frame S and S′. Hence at that instant,

$t = 0$ for the frame S and

$t' = 0$ for the frame S′

An event occurs at the point P in space. It is described by the co-ordinates (x,y,z,t) relative to the frame S and by the co-ordinates (x',y',z',t') relative to the frame S′ (Fig 1.3).

For the observer in the frame S the origin O′ moves from O to O′ in time t with velocity V.

$$\therefore V = \frac{OO'}{t} \qquad \therefore OO' = Vt.$$

Let PM be perpendicular to X and X'-axes. Then by definition of co-ordinates, $x = OM$ and $x' = O'M$.

$$\text{Now, } OM = OO' + O'M$$
$$\therefore x = Vt + x'$$
$$\therefore x' = x - Vt.$$

Since there is no relative motion along Y and Z-axes and since X and X' - axes are coinciding, $y' = y$, $z' = z$

According to Newton, time is absolute by its very nature flowing uniformly without reference to anything external. Hence there is one universal time for all frames of reference. i.e. the event must be occuring at the same time relative to both the frames of reference.

$$\therefore t' = t.$$

The equations,

$$\boxed{\begin{aligned} x' &= x - Vt \\ y' &= y,\ z' = z \\ t' &= t \end{aligned}} \qquad \text{............(1.1)}$$

are called Galilean transformation equations or classical transformation equations for space and time co-ordinates.

1.6 INVARIANCE OF A PHYSICAL QUANTITY AND A PHYSICAL LAW :

Definition : A physical quantity is said to be invariant if its value remains the same when the co-ordinate transformations are carried out from one frame of reference to another frame of reference.

For example, we can see from equations (1.1) that Y and Z co-ordinates are invariant, time also is invariant but X co-ordinate is not invariant under Galilean transformations.

Definition : A physical law is said to be invariant if the equation representing the law retains its form when the co-ordinate transformations are carried out from one frame of reference to another frame of reference.

The concept of invariance is going to play a very important role in Relativity. In what follows we shall investigate whether several quantities and several laws belonging to Mechanics and Electrodynamics are invariant or not under Galilean transformations of Newtonian Relativity.

GALILEAN TRANSFORMATIONS IN MECHANICS

1.7 INVARIANCE OF SPACE :

Let us consider two points in space. Relative to the frame S, their co-ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then the distance between the two points as measured in the frame S is given by

$$s = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Let the co-ordinates of the two points relative to the frame S' be (x_1', y_1', z_1') and (x_2', y_2', z_2') respectively, Then the distance between the two points as measured in the frame S' is given by

$$s' = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2 + (z_2' - z_1')^2} \qquad ...(1.2)$$

Now, according to Galilean transformations ,

$$\boxed{x' = x - Vt .} \quad \text{— Formula}$$

$\therefore$ For the first point, $\quad x_1' = x_1 - Vt_1.$

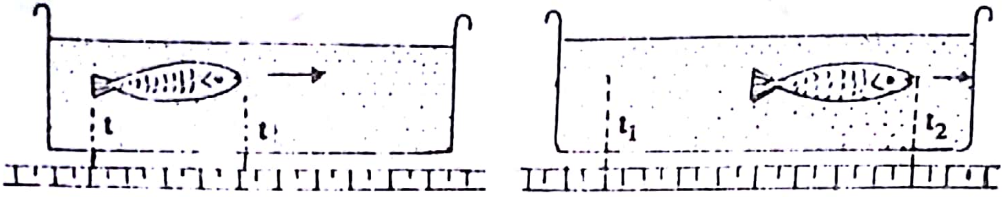
For the second point, $\quad x_2' = x_2 - Vt_2.$

Whenever the length of an object (particularly a moving object) is to be measured the end points of the object should be located on the scale (or on the axis) simultaneously i.e. at the same instant t. Hence $t_1 = t_2 = t$

For example, you want to measure the length of a moving fish. If you would locate its tail and head on the scale at the same instant t [Fig. 1. 4 (a)] you will get the length of the fish. If you would locate its tail at an instant t_1 and its head at some later instant t_2 [Fig. 1.4 (b)] can you get the length of the fish?

$$\therefore x_1' = x_1 - Vt \quad \text{and} \quad x_2' = x_2 - Vt.$$

$$\therefore x_2' - x_1' = (x_2 - Vt) - (x_1 - Vt) = x_2 - x_1 \qquad \text{....(1.3)}$$



(a) (b)

Fig. 1.4 : Measurement of the length of a moving fish

According to Galilean transformations, $y' = y$

$$\therefore y_1' = y_1 \quad \text{and} \quad y_2' = y_2$$

$$\therefore y_2' - y_1' = y_2 - y_1 \qquad \text{.....(1.4)}$$

Similarly, $z_2' - z_1' = z_2 - z_1$ .....(1.5)

By equations (1.3), (1.4) and (1.5), equation (1.2) becomes,

$$s' = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = s$$

Thus the distance between two points is invariant under Galilean transformations. //

1.8 INVARIANCE OF TIME - INTERVALS :

Let us consider two events occuring in space. Relative to the frame S, they occur at the instants t_1 and t_2 respectively.

Time - interval between the two events as measured in the frame S is given by

$$\Delta t = t_2 - t_1$$

Let the same events occur at the instants t_1' and t_2' respectively relative to the frame S'.

Time - interval between the two events as measured in the frame S' is given by

$$(\Delta t)' = t_2' - t_1'.$$

According to Galilean transformations, $t' = t$.

$\therefore$ For the first event, $t'_1 = t_1$.

For the second event, $t'_2 = t_2$.

$$\therefore (\Delta t)' = t'_2 - t'_1 = t_2 - t_1 = \Delta t.$$

Thus the time - interval between the two events is invariant under Galilean transformations.

1.9 INVARIANCE OF MASS :

In Mechanics we measure the mass of a body by finding the ratio of the force acting on the body to the acceleration thereby acquired by the body. This mass is a measure of inertia of the body and hence it is called inertial mass of the body. Mass is also a measure of the quantity of matter contained by the body. Hence it is the intrinsic property of the body alone. It does not depend upon the motion of the body or of the observer or on anything external. According to the principle of conservation of mass, a basic principle of Chemistry and Classical Physics the mass of an isolated system remains constant eternally. Hence there is one and only one universal mass of each body. Therefore the mass of a body must remain the same in both the frames S and S'.

i.e. $m' = m$.

Thus mass is invariant under Galilean transformations.

1.10 GALILEAN TRANSFORMATIONS OF VELOCITY :

Let us consider a particle moving in space. Let its velocity relative to the frame S be u having components u_x, u_y, u_z along X, Y and Z-axes respectively.

$$\text{Then } u_x = \frac{dx}{dt}, \quad u_y = \frac{dy}{dt}, \quad u_z = \frac{dz}{dt}$$

Let its corresponding velocity relative to the frame S' be u' having components u'_x, u'_y, u'_z along X', Y', Z'-axes respectively.

$$\text{Then } u'_x = \frac{dx'}{dt'}, \quad u'_y = \frac{dy'}{dt'}, \quad u'_z = \frac{dz'}{dt'}$$

$$\text{Now, } u'_x = \frac{dx'}{dt'} = \frac{d}{dt}(x - Vt) = \frac{dx}{dt} - \frac{d}{dt}(Vt) = u_x - V$$

(Please note that V is constant for inertial frames).

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{dt} = u_y$$

$$u'_z = \frac{dz'}{dt'} = \frac{dz}{dt} = u_z$$

Thus we have,
$$\boxed{\begin{aligned} u'_x &= u_x - V \\ u'_y &= u_y \\ u'_z &= u_z \end{aligned}} \qquad .....(1.6)$$

These are the Galilean transformation equations for velocity components.

From these equations we can see easily that Y and Z-components of velocity are invariant but X-component is not invariant under Galilean transformations. i.e. the velocity of a particle is different in different frames of reference moving relative to each other.

1.11 GALILEAN TRANSFORMATIONS OF ACCELERATION :

Let the acceleration of a particle relative to the frame S be a having components a_x, a_y, a_z along X, Y and Z -axes respectively.

Then $a_x = \frac{du_x}{dt}$, $a_y = \frac{du_y}{dt}$, $a_z = \frac{du_z}{dt}$

Let the corresponding acceleration of the particle relative to the frame S′ be a' having components a'_x, a'_y, a'_z along X′, Y′ and Z′ - axes respectively.

Then $a'_x = \frac{du'_x}{dt'}$, $a'_y = \frac{du'_y}{dt'}$, $a'_z = \frac{du'_z}{dt'}$,

Now, $$a'_x = \frac{du'_x}{dt'} = \frac{d}{dt}(u_x - V) = \frac{du_x}{dt} - \frac{dV}{dt} = \frac{du_x}{dt} - 0 = a_x$$

$$a'_y = \frac{du'_y}{dt'} = \frac{du_y}{dt} = a_y$$

$$a'_z = \frac{du'_z}{dt'} = \frac{du_z}{dt} = a_z$$

Thus we have,
$$\boxed{\begin{aligned} a'_x &= a_x \\ a'_y &= a_y \\ a'_z &= a_z \end{aligned}} \qquad \text{.....(1.7)}$$

Hence we can say, $a' = a$ i.e. the acceleration of the particle is invariant under Galilean transformations.

1.12 GALILEAN TRANSFORMATIONS OF FORCE :

By Newton's second law of motion, the force is given by

$$F = ma.$$

Relative to the frame S, $F = ma.$

Relative to the frame S′, $F' = m'a'$

But $m' = m$ and $a' = a$. (see sections § 1.9 and § 1.11)

$\therefore F' = F$

Thus force is invariant under Galilean transformations.

1.13 INVARIANCE OF NEWTON'S LAWS OF MOTION :

Since $F' = F$, $m' = m$, $a' = a$ the equation

$F = ma$

has the same form in the frames S and S′. Hence Newton's laws of motion are invariant under Galilean Transformations.

1.14 INVARIANCE OF THE PRINCIPLE OF CONSERVATION OF MOMENTUM :

The momentum of a particle having mass m and moving with a velocity u is given by

$p = mu.$

We have seen in § 1.9 and § 1.10 that mass is invariant but velocity is not invariant under Galilean transformations. Hence the momentum of the particle is not invariant under Galilean transformations. It will be interesting to verify whether the Principle of conservation of momentum is invariant or not. Let us investigate this.

Consider a head-on collision between two particles having masses m_1 and m_2 respectively. Before the collision, they are moving with velocities u_1 and u_2 respectively in the direction of *X-axis*. After the collision they are moving with velocities v_1 and v_2 respectively in the same direction. By Principle of conservation of momentum for the frame S we get,

$$\left.\begin{array}{l}\text{Total initial momentum}\\ \text{of the system}\end{array}\right\} = \left\{\begin{array}{l}\text{Total final momentum}\\ \text{of the system}\end{array}\right.$$

i.e. $m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$ .....(1.8)

We shall transform this equation to the frame S' as follows :

Since mass is invariant, $m'_1 = m_1,\ m'_2 = m_2$

By Galilean transformations of velocity,

$$u'_x = u_x - V \qquad \therefore u_x = u'_x + V.$$

For the first particle, $u_1 = u'_1 + V,\quad v_1 = v'_1 + V$

For the second particle, $u_2 = u'_2 + V,\quad v_2 = v'_2 + V$

Substituting these in equation (1.8),

$$m'_1(u'_1 + V) + m'_2(u'_2 + V) = m'_1(v'_1 + V) + m'_2(v'_2 + V)$$

$$\therefore m'_1u'_1 + m'_1V + m'_2u'_2 + m'_2V = m'_1v'_1 + m'_1V + m'_2v'_2 + m'_2V$$

$$\therefore m'_1u'_1 + m'_2u'_2 = m'_1v'_1 + m'_2v'_2 \quad .....(1.9)$$

The equations (1.8) and (1.9) have the same from. Hence the Principle of conservation of momentum is invariant under Galilean transformations.

Concluding Remarks :

Upto this point we have got all pleasant results matching with our expectations. The fundamental quantities of Mechanics, viz. space, time and mass are invariant under Galilean transformations.i.e. space, time and mass are absolute quantities in Newtonian Relativity. Their values remain the same in all inertial frames of reference. Even the fundamental laws of Mechanics, viz. Newton's laws of motion, conservation laws etc. are invariant. i.e. the same laws are valid in all inertial frames of reference and for different inertial frames we do not have to formulate separate laws of Mechanics. These conclusions have already been tested experimentally in Classical Mechanics.

GALILEAN TRANSFORMATIONS IN ELECTRODYNAMICS

1.15 HISTORICAL SURVEY OF ELECTRODYNAMICS :

Upto the first half of nineteenth century, Mechanics was the most dominating branch of Physics. Mechanical views were used to explain various phenomena in all other branches of Physics. For example, kinetic theory of gases was formulated to explain various phenomena of heat. Wave theory was formulated to explain propagation of sound and light. Even corpuscular theory of light was manifestation of these mechanical views. Attractions and repulsions by electric charges and magnetic poles were explained by Coulomb's law using the same mechanical views.

During the second half of nineteenth century, Faraday introduced the concept of "field" into Physics in order to explain "action at a distance" from electric charges and magnetic poles. According to this concept, the electric charges and magnetic poles do not exert the forces of attraction or repulsion directly on other electric charges and magnetic poles respectively. An electric charge produces an electric field around itself which exerts the forces on other charges lying in the field. Similarly a magnetic pole produces a magnetic field around itself which exerts forces on other magnetic poles lying in the field.

It was observed experimentally that electric field and magnetic field are interconnected. Oersted's experiment of deflection of a magnetic needle kept in the vicinity of a wire carrying a current lead us to conclude that a changing electric field gives rise to a magnetic field. Faraday's experiments of induced currents due to the movement of a magnet lead us to conclude that a changing magnetic field gives rise to an electric field. Hence instead of treating an electric field and a magnetic field as separate entities, both were united into one entity, the electromagnetic field. The behaviour of an electromangetic field is studied in a new branch of Physics named as Electromagnetism. Here the laws of Mechanics are not applicable. In 1862 Maxwell formulated the laws and the equations of electromagnetism.

When a disturbance is produced at a point in an electromagnetic field it induces a change in the field surrounding the point. This leads to a change further away. Thus the original disturbance has a tendency to spread in space with some velocity. The static electromagnetic field becomes dynamic and electromagnetic waves are produced in electromagnetic field.

According to Maxwell's theory of electrodynamics these electromagnetic waves travel with a velocity $c = \frac{1}{\sqrt{\epsilon\mu}}$ where ϵ is permittivity and μ is permeability of the transmitting medium. Thus the velocity of the electromagnetic waves depends upon the electromagnetic property of the transmitting medium. In the absence of the medium, i.e. in empty space, the velocity of the electromagnetic waves is given by

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$= \frac{1}{\sqrt{\frac{1}{4\pi \times 9 \times 10^9} \times 4\pi \times 10^{-7}}}$$

$$= \sqrt{9 \times 10^{16}} = 3 \times 10^8 \text{ m/s.}$$

Thus Maxwell's theory of electrodynamics leads us to conclude that the electromagnetic waves travel in empty space with an uniform and constant velocity, $c = 3 \times 10^8$ m/s in all directions. Now, even light waves are found to travel with the same velocity in empty space. Hence Maxwell considered light to be consisting of electromagnetic waves.

In 1890, when Hertz could produce electromagnetic waves experimentally, Maxwell's electrodynamics became a well established branch of Physics, different from Classical Mechanics.

1.16 THE ETHER HYPOTHESIS :

Even after introducing the new electromagnetic wave theory, the physicists of that time were accustomed to think in terms of the old mechanical views. They stressed that every wave must have a material medium in which it would travel. For example, sound waves travel in air, ocean waves travel in water. Light waves also must have some medium. It was known that light can travel through vacuum also. Hence Maxwell and Hertz themselves introduced a hypothetical material medium named as "ether". The light waves and all other electromagnetic waves were assumed to travel through ether. Ether was completely an imaginary medium. It was considered to be perfectly transparent and penetrating all matter in the universe. It had no mass, zero density and no effect of gravity. It never exerted any frictional resistance to the matter moving through it. It was undetectable by any means and by all means. In short, it had no property except to transmit light and other electromagnetic waves with velocity $c = 3 \times 10^8$ m/s.

At the end of nineteenth century the ether hypothesis was regarded as the most promising and absolutely essential hypothesis.

Are the Maxwell's laws of electrodynamics invariant under Galilean transformations? i.e. Do Maxwell's equations which represent these laws, retain their form under Galilean transformations? In what follows we shall investigate whether the wave equation of electromagnetic waves (which is derived from Maxwell's equations) is invariant under Galilean transformations.

1.17 NON-INVARIANCE OF THE WAVE EQUATION OF ELECTRO-DYNAMICS UNDER GALILEAN TRANSFORMATIONS :

The electromagnetic wave equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

$$\frac{\partial^2\phi}{\partial s^2} = \frac{1}{c^2}\frac{\partial\phi}{\partial t}$$

where ϕ is scalar potential which is a function of x, y, z, and t.

i) Using the chain rule of partial derivatives,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\cdot\frac{\partial x'}{\partial x}+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial y'}{\partial x}+\frac{\partial\phi}{\partial z'}\cdot\frac{\partial z'}{\partial x}+\frac{\partial\phi}{\partial t'}\cdot\frac{\partial t'}{\partial x}$$

Now, $$\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x}(x - Vt) = \frac{\partial x}{\partial x} - V\frac{\partial t}{\partial x} = 1 - V\times 0 = 1$$

$$\frac{\partial y'}{\partial x} = \frac{\partial y}{\partial x} = 0\,, \qquad \frac{\partial z'}{\partial x} = \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial t'}{\partial x} = \frac{\partial t}{\partial x} = 0$$

Substituting these values, the initial equation becomes

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\times 1+\frac{\partial\phi}{\partial y'}\times 0+\frac{\partial\phi}{\partial z'}\times 0+\frac{\partial\phi}{\partial t'}\times 0 = \frac{\partial\phi}{\partial x'}$$

$$\therefore \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad \text{...(1.10)}$$

ii) $$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x'}\cdot\frac{\partial x'}{\partial y}+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial y'}{\partial y}+\frac{\partial\phi}{\partial z'}\cdot\frac{\partial z'}{\partial y}+\frac{\partial\phi}{\partial t'}\cdot\frac{\partial t'}{\partial y}$$

Now, $$\frac{\partial x'}{\partial y} = \frac{\partial}{\partial y}(x - Vt) = \frac{\partial x}{\partial y} - V\frac{\partial t}{\partial y} = 0$$

$$\frac{\partial y'}{\partial y} = \frac{\partial y}{\partial y} = 1\,, \qquad \frac{\partial z'}{\partial y} = \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial t'}{\partial y} = \frac{\partial t}{\partial y} = 0$$

$$\therefore \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x'}\times 0+\frac{\partial\phi}{\partial y'}\times 1+\frac{\partial\phi}{\partial z'}\times 0+\frac{\partial\phi}{\partial t'}\times 0$$

$$= \frac{\partial\phi}{\partial y'} \qquad \therefore \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \qquad \text{...(1.11)}$$

iii) Using the same method as in (ii) we can prove,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad ...(1.12)$$

iv) $$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x'} \cdot \frac{\partial x'}{\partial t} + \frac{\partial \phi}{\partial y'} \cdot \frac{\partial y'}{\partial t} + \frac{\partial \phi}{\partial z'} \cdot \frac{\partial z'}{\partial t} + \frac{\partial \phi}{\partial t'} \cdot \frac{\partial t'}{\partial t}$$

Now, $$\frac{\partial x'}{\partial t} = \frac{\partial}{\partial t}(x - Vt) = \frac{\partial x}{\partial t} - V\frac{\partial t}{\partial t} = -V$$

$$\frac{\partial y'}{\partial t} = \frac{\partial y}{\partial t} = 0, \qquad \frac{\partial z'}{\partial t} = \frac{\partial z}{\partial t} = 0$$

$$\frac{\partial t'}{\partial t} = \frac{\partial t}{\partial t} = 1$$

$$\therefore \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x'} \cdot (-V) + \frac{\partial \phi}{\partial y'} \times 0 + \frac{\partial \phi}{\partial z'} \times 0 + \frac{\partial \phi}{\partial t'} \times 1$$

$$= \frac{\partial \phi}{\partial t'} - V\frac{\partial \phi}{\partial x'} \qquad \therefore \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'} \qquad ...(1.13)$$

The electromagnetic wave equation in the frame S is as follows :

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0 \qquad ...(1.14)$$

$$\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = 0$$

By equations (1.10), (1.11), (1.12) & (1.13) this equation becomes

$$\left[\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\left(\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right)^2\right]\phi = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} - \frac{1}{c^2}\left(\frac{\partial^2}{\partial t'^2} - 2V\frac{\partial^2}{\partial t'.\partial x'} + V^2\frac{\partial^2}{\partial x'^2}\right)\right]\phi = 0$$

$$\therefore \frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2} + \frac{2V}{c^2}\frac{\partial^2 \phi}{\partial t'.\partial x'} - \frac{V^2}{c^2}\frac{\partial^2 \phi}{\partial x'^2} = 0 \qquad ...(1.15)$$

This is the wave equation in the frame S'. Due to the presence of the last two terms, the equation(1.15) is not having the same form as the equation (1.14). Hence the electromagnetic wave equation is not invariant under Galilean transformations.

Remarks :

i) Can you think of the implications of the above result? In the frame S the electromagnetic waves propagate in space according to one equation and in the frame S′ the same waves appear to propagate according to some other equation! i.e. in the frame S, Maxwell's equations are obeyed but in the frame S', they are not obeyed! This means that for different inertial frames of reference we shall have to formulate different laws of electrodynamics!

ii) In case, if the velocity (V) of the frame S' relative to the frame S is much smaller than the velocity (c) of light then the last two terms of equation (1.15) become very small and can be neglected in comparison with other terms. Then the form of the equation (1.15) would be the same as that of (1.14). Thus if the frame S' is moving with a very small velocity then the wave equation becomes invariant under Galilean transformations.

iii) Just look at the equations (1.10), (1.11), (1.12) and (1.13). You will observe that the wave equation is not invariant due to the manner in which the time co-ordinate is transformed. There may be something wrong in our concept of time. As we shall proceed further you will realize this.

1.18 VELOCITY OF PROPAGATION OF LIGHT AND GALILEAN TRANSFORMATIONS:

As we have seen earlier, Maxwell's electrodynamics incorporates the waves of light into its fold. According to Maxwell's theory, light consists of electromagnetic waves which travel in empty space (i.e. through ether) with an uniform, constant velocity $c = 3 \times 10^8$ m/s in all directions. This has been proved not only theoretically but has been verified experimentally also.

Now let us attach the frame S to ether. A ray of light is travelling along X-axis with velocity c relative to the frame S. Hence $u_x = c$. By Galilean transformations of velocity,

$$u'_x = u_x - V,$$

$$u'_y = u_y,$$

$$u'_z = u_z.$$

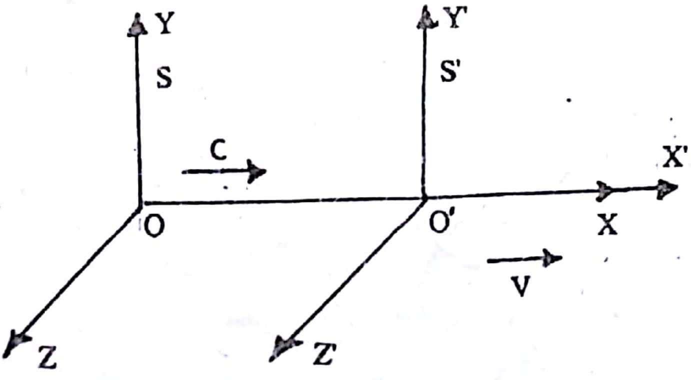


Fig. 1.5 Velocity of light in frames S & S'

Hence the velocity of the ray relative to the frame S' is given by

$$u'_x = c - V$$

If the ray travels in X-Y plane in different directions with velocity c then its velocity varies from $c - V$ to $c + V$ relative to the frame S′. Thus the velocity of light is not invariant under Galilean transformations. i.e. for different observers in different states of motion the velocity of light comes out to be different. This contradicts the conclusion of Maxwell's theory of electrodynamics that the velocity of light and other electromagnetic waves depends only upon the electromagnetic property of the transmitting medium and in vacuum it is always equal to $c = 3 \times 10^8$ m/s in all directions and for all observers.

1.19 ABSOLUTE FRAME OF REFERENCE :

Physicists of late 19th Century observed that although the laws of mechanics are invariant under Galilean transformations, the laws of electrodynamics are not invariant under Galilean transformations. For different frames, different laws are to be devised. Even the velocity of light varies from one frame of reference to another. Hence they assumed that there existed one unique, universal frame of reference named as "absolute frame of reference". In this frame of reference the laws and the equations of electrodynamics would take their standard form of Maxwell's theory. Only in this frame of reference the velocity of light would be $c = 3 \times 10^8$ m/s in empty space. In other frames moving relative to this absolute frame, the velocity of light would be different from c. Since the absolute frame of reference enjoys all these privileges of electrodynamics it is also called a privileged frame of reference.

In sound, whenever we find the velocity of sound it is always relative to the stationary air through which the sound waves travel. Using this analogy, the velocity of light is taken as $c = 3 \times 10^8$ m/s relative to the imaginary transmitting medium, the ether. Hence this absolute frame is attached to ether. Thus the ether becomes absolute frame of reference.

Now since the earth travels in space through ether, relative to this absolute frame of reference, the velocity of light measured relative to the earth must be different from $c = 3 \times 10^8$ m/s. In order to detect the effect of earth's motion on the velocity of light, A.A. Michelson and E.W. Morley in 1887 devised one epoch-making experiment which is described in what follows.

1.20 MICHELSON - MORLEY EXPERIMENT :

Aim : Let us assume that the velocity of light is not the same in all frames of reference and only in absolute frame of reference (or privileged frame of reference) attached to ether the velocity of light is

$c = 3 \times 10^8$ m/s in empty space. In all other frames moving relative to this absolute frame the velocity of light would be different from c.

Earth is moving in space through ether relative to this absolute frame. It rotates about its own axis and it orbits around the sun. Even if we neglect earth's spinning motion its orbital velocity around the sun is $V = 30$ km/s = 30000 m/s. which is comparable to the velocity of light.

Thus the earth moves with a velocity V relative to the absolute frame and light rays are travelling with velocity c relative to the same absolute frame. Hence when the velocity of light is measured relative to the earth it would be different from c, according to Galilean transformations of velocity (see Fig. 1.5). In order to test whether the velocity of light measured relative to the earth remains the same as c or changes, Michelson and Morley in 1887 performed this experiment using Michelson's interferometer.

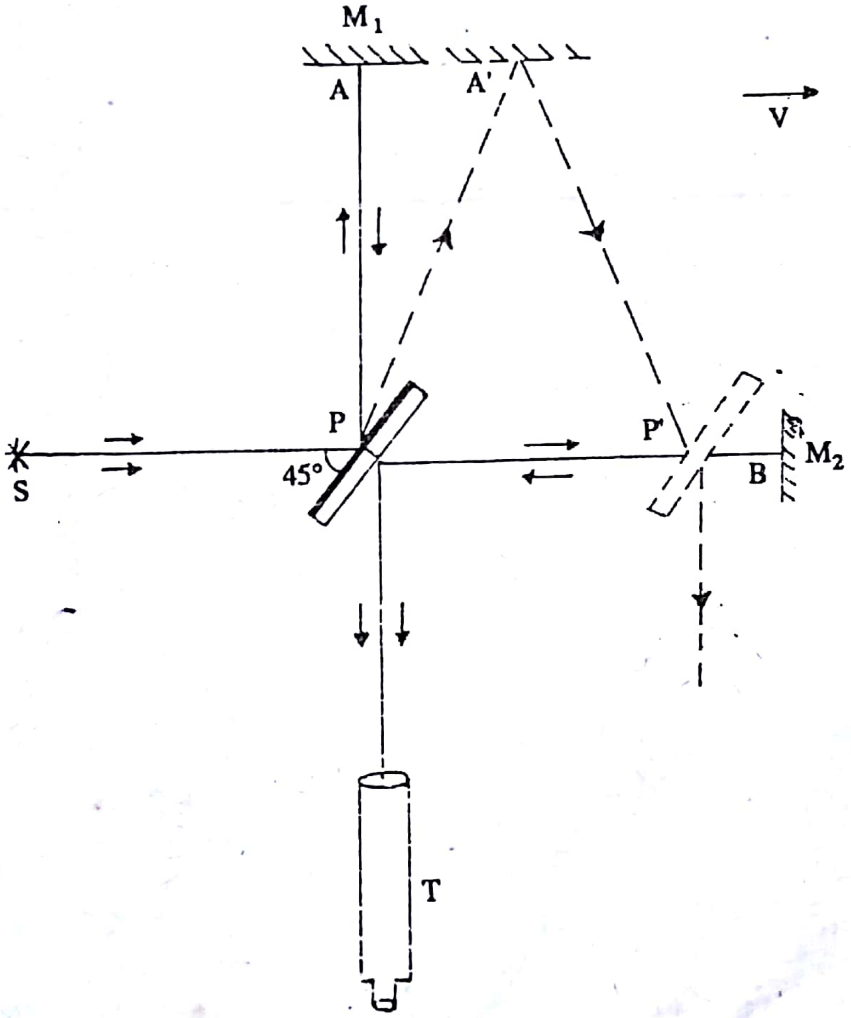


Fig. 1.6 : Schematic arrangement of the interferometer

A beam of monochromatic light (sodium light) of wavelength λ from the source S is incident to a semisilvered glass plate P placed at an angle of 45° with the direction of the incident beam. The beam is partly reflected and partly transmitted in two orthogonal directions. The reflected part of the beam travels in the direction perpendicular to the incident beam and falls normally on the mirror M_1. It is then reflected back to P. The transmitted part of the beam travels in the direction of the incident beam and falls normally on the mirror M_2. Then it is reflected back to P. The rays returning to P are reflected and refracted from P and travel towards the telescope T. Along the path the rays interfere with each other and interference pattern can be seen in the telescope T. The mirrors M_1 and M_2 are so adjusted that PA = PB = l, say.

Method : The whole apparatus is moving along with the earth through ether with velocity V. Let the direction of the incident beam SP coincide with the direction of motion of the earth through ether. (See Fig. 1.6)

Since the earth is moving relative to ether, the rays reflected from P strike the mirror M_1 not at A but at A' and are reflected along A'P' to reach the plate at P'. Since the velocity V is uniform, PA' = A'P'.

Let the time required by the reflected rays to reach the mirror M_1 be t.

Then $c = \frac{PA'}{t}$ $\qquad \therefore PA' = c.t.$

The mirror M_1 moves from A to A' with velocity V in the same time t. Hence $V = \frac{AA'}{t}$ and $AA' = Vt$

In right angled $\Delta PAA'$,

$$PA'^2 = PA^2 + AA'^2$$

$$\therefore c^2t^2 = l^2 + V^2t^2$$

$$\therefore t^2(c^2 - V^2) = l^2$$

$$\therefore t^2 = \frac{l^2}{c^2 - V^2} \qquad \therefore t = \frac{l}{\sqrt{c^2 - V^2}}$$

Since PA' = A'P', the time required to travel A'P' is $t = \frac{l}{\sqrt{c^2 - V^2}}$

Hence total time required by the reflected rays to come back to the plate is given by

$$t_1 = \frac{l}{\sqrt{c^2 - V^2}} + \frac{l}{\sqrt{c^2 - V^2}} = \frac{2l}{\sqrt{c^2 - V^2}}$$

The velocity of the transmitted rays travelling towards M_2 relative to the apparatus (i.e. relative to the earth) is $c - V$. The velocity of the rays reflected from M_2 relative to the apparatus is $c + V$. These rays travel through distance l in each direction. Hence the total time taken by the transmitted rays to come back to the plate is given by

$$t_2 = \frac{l}{c - V} + \frac{l}{c + V} = \frac{lc + lV + lc - lV}{c^2 - V^2} = \frac{2lc}{c^2 - V^2}$$

∴ Time difference between the rays returning to the plate P is given by

$$\Delta t = t_2 - t_1 = \frac{2lc}{c^2 - V^2} - \frac{2l}{\sqrt{c^2 - V^2}}$$

$$= \frac{2lc}{c^2\left(1 - \frac{V^2}{c^2}\right)} - \frac{2l}{c\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{2l}{c}\left[\frac{1}{1 - \frac{V^2}{c^2}} - \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\right]$$

$$= \frac{2l}{c}\left[\left(1 - \frac{V^2}{c^2}\right)^{-1} - \left(1 - \frac{V^2}{c^2}\right)^{-1/2}\right]$$

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By Binomial Theorem,

$$(a + x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \ldots..$$

$$\therefore \left(1 - \frac{V^2}{c^2}\right)^{-1} = (1)^{-1} + (-1)(1)^{-2}\left(-\frac{V^2}{c^2}\right) + \frac{(-1)(-2)}{2}(1)^{-3}\left(-\frac{V^2}{c^2}\right)^2 +$$

$$= 1 + \frac{V^2}{c^2} + \left(\frac{V^2}{c^2}\right)^2 + \text{ terms in higher powers of } \frac{V^2}{c^2}$$

Now, V is small in comparison with c. Hence $\frac{V}{c}$ is much less than 1. Due to this, the terms in higher powers of $\frac{V}{c}$ become negligible in comparison with the first two terms on R.H.S.

$$\therefore \left(1 - \frac{V^2}{c^2}\right)^{-1} = 1 + \frac{V^2}{c^2}$$

Similarly,

$$\left(1 - \frac{V^2}{c^2}\right)^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right)(1)^{-3/2}\left(-\frac{V^2}{c^2}\right) + \ldots..$$

$$= 1 + \frac{1}{2}\frac{V^2}{c^2}$$

$$\therefore \Delta t = \frac{2l}{c}\left[\left(1+\frac{V^2}{c^2}\right)-\left(1+\frac{V^2}{2c^2}\right)\right]$$

$$= \frac{2l}{c}\left[1+\frac{V^2}{c^2}-1-\frac{V^2}{2c^2}\right]$$

$$= \frac{2l}{c}\left[\frac{1}{2}\frac{V^2}{c^2}\right] = \frac{lV^2}{c^3}$$

$\therefore$ Path difference between the rays returning to the plate P is given by

$$\Delta x = c.\Delta t = \frac{lV^2}{c^2} \text{ metre} = \frac{lV^2}{c^2\lambda} \text{ wavelengths.}$$

Corresponding to this path difference the interference pattern is observed in the telescope T.

Then the whole apparatus is rotated through 90° in the plane of the above diagram (Fig. 1.6) in the clockwise sense. Now the rays reflected from the plate P travel in the direction parallel to the velocity V and the rays transmitted from the plate P travel in the direction perpendicular to the velocity V.

$\therefore$ Time taken by the reflected rays to come back to the plate

$$= t'_1 = t_2$$

Time taken by the transmitted rays to come back to the plate

$$= t'_2 = t_1$$

$\therefore$ Time difference between the rays returning to the plate P is given by

$$(\Delta t)' = t'_2 - t'_1 = t_1 - t_2 = -(t_2 - t_1) = -\Delta t.$$

Path difference between the rays returning to the plate P is given by

$$(\Delta x)' = c.(\Delta t)' = c.(-\Delta t) = -\Delta x = -\frac{lV^2}{c^2\lambda} \text{ wavelengths.}$$

Thus on rotating the apparatus through 90° the path difference changes from Δx to $(\Delta x)'$. The shift in the path difference is

$$= \Delta x - (\Delta x)' = \frac{lV^2}{c^2\lambda} - \left(-\frac{lV^2}{c^2\lambda}\right) = \frac{2lV^2}{c^2\lambda} \text{ wavelengths.}$$

Corresponding shift in the interference pattern

$$= \Delta N = \frac{2lV^2}{c^2\lambda} \text{ fringes.}$$

Thus on rotating the apparatus through 90° the interference pattern must be displaced through $\Delta N = \frac{2lV^2}{c^2\lambda}$ fringes. Substituting the values of l and λ taken in the experiment and the known values of V and c this shift was calculated to be $\Delta N = 0.37$ fringe. (i.e. almost one-third of a fringe).

Result : The instrument used in the experiment was so accurate that a displacement of $\frac{1}{100}$ of the about amount (i.e. 0.0037 fringe) could have been observed if it would have occured. But Michelson and Morley could not observe any significant displacement in the inference pattern i.e. $\Delta N = 0$. They repeated the experiment at various times of the day and in different seasons of the year but the result was always in negative,

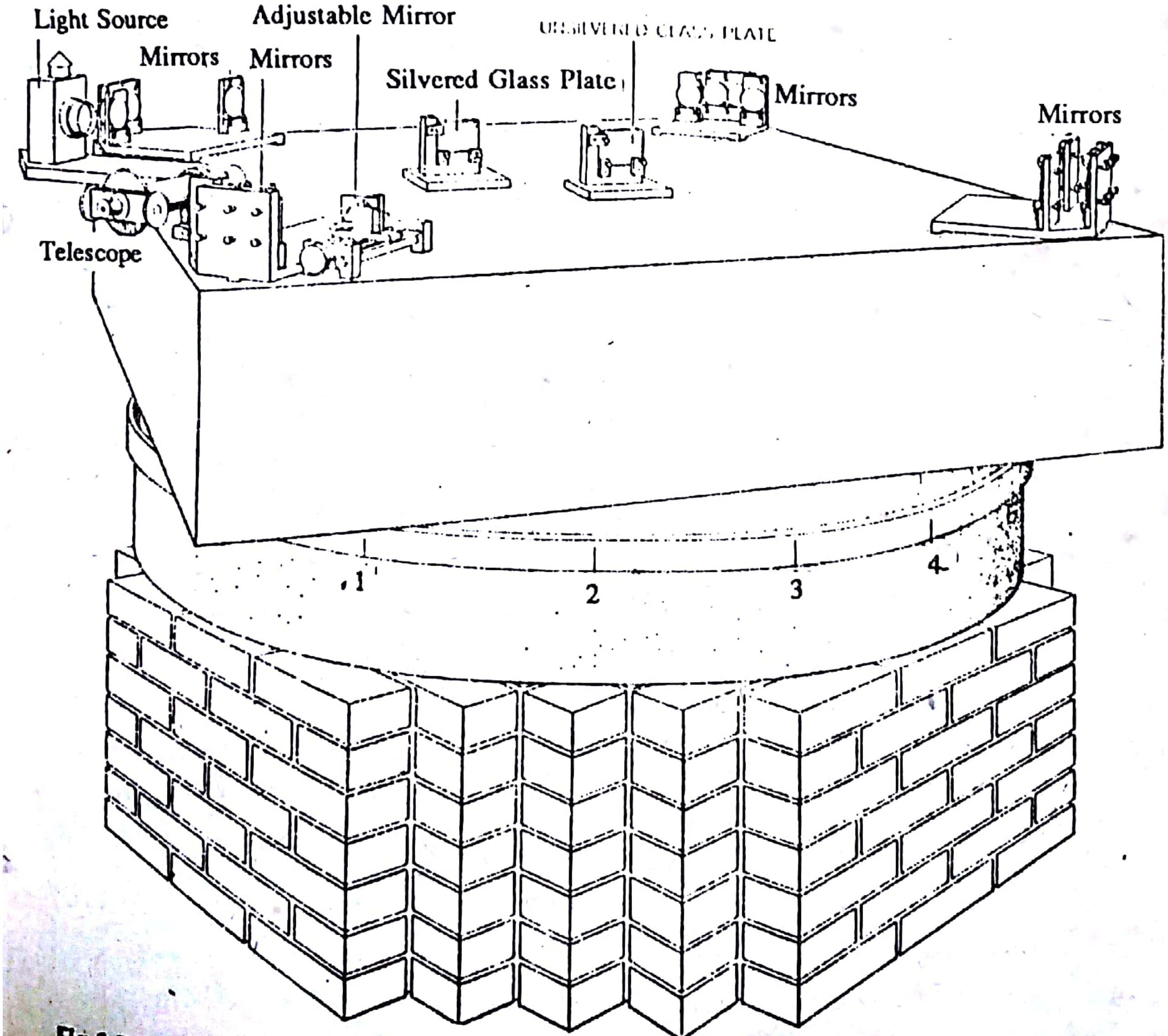

Fig 1.7 Michelson and Morley had mounted the interferometer on a massive stone slab which could be kept rotating continuously.

1.21 DISCUSSION ON NEGATIVE EXPERIMENT:

Thus the attempt made by Michelson and Morely to detect the effect of the earth's motion on the velocity of light failed miserably. Truely speaking, this happened due to the absurdity of our initial assumption that velocity of light is $c = 3 \times 10^8$ m/s only in absolute frame of reference attached to ether and in all other frames moving relative to this absolute frame the velocity of light would be different from c. If you remember, on this assumption we had developed the theory of the experiment and we had expected some result. The expected result could not be obtained in the experiment because our initial assumption was wrong.

But physicists of late nineteenth century were not thinking in this manner. They were not ready to give up the idea of an absolute frame of reference attached to ether. They put forward several hypotheses which would preserve the concept of unique absolute frame attached to ether and still explain the null result obtained by Michelson and morley. Among these hypotheses, the most prominent were (i) Ether-drag hypothesis and (ii) Lorentz-Fitzgerald contraction hypothesis.

1.22 THE ETHER - DRAG HYPOTHESIS :

The experimental result of Michelson-Morley experiment implies that,

$$\text{shift in the interference pattern} = \Delta N = 0.$$

$$\text{i.e.} \quad \frac{2lV^2}{c^2\lambda} \text{ fringe} = 0.$$

Now c is not every large in comparison with V i.e. it is not infinite. Hence the above equation reduces to

$$V = 0.$$

i.e. the velocity of the earth relative to ether is zero. This means that the earth is stationary relative to ether. But we know that the earth is moving in space. Hence while moving in space the earth must be dragging the ether along with itself and due to this its velocity relative to ether is zero. This is called ether-drag hypothesis.

Due to ether-drag the velocity of light relative to the earth would be the same as its velocity relative to ether i.e. $c = 3 \times 10^8$ m/s. Hence there is no effect of the earth's motion on the velocity of light. Thus the ether-drag hypothesis can account for the negative result of Mischelson - Morley experiment.

We shall see in the next section that if we accept the ether-drag hypothesis then the experimentally observed phenomenon of 'aberration of light' can not be explained. Hence ether-drag hypothesis was dropped ultimately.

1.23 ABERRATION OF LIGHT :

In astronomical observations made in eighteenth century, it was found that a star lying directly overhead (i.e. at the Zenith) appears to move in a circle during a period of one year. The diameter of the circle subtends an angle of 41" at the earth. Stars in other positions appear to move along elliptical orbits but with the same period of one year. This phenomenon is called Aberration of light or Steller aberration and it can be understood as follows:

We consider the rays of light emitted by the star and travelling in vertically downward direction towards the earth, through ether. If the earth were stationary relative to the star, the star could have been seen through the telescope by keeping the telescope vertical [See Fig. 1.8 (a)]. But the earth is moving relative to the star, say towards right. Hence by the time the light rays travel in the telescope (from its objective to eyepiece) the telescope gets shifted towards right and the rays strike the walls of the telescope. As a result of this, the star cannot be seen through the telescope if we keep the telescope vertical.

When we keep the telescope inclined at an angle α with the vertical direction of the rays, [Fig. 1.8 (b)] by the time Δt during which the light rays travel in the telescope, the telescope moves through a distance $V.\Delta t$ (where V is the velocity of the earth relative to the star) to right and the eyepiece comes exactly below the star. Hence the star can be seen through the telescope. In time Δt, the rays travel through a distance $c.\Delta t$ in vertically downward direction. As shown in Fig. 1.8(b),

$$\tan\alpha = \frac{V.\Delta t}{c.\Delta t} = \frac{V}{c} \qquad ...(1.16)$$

Substituting the values of V and c, α comes out to be 20.5".

When the earth completes one revolution around the Sun during one year, the axis of the telescope directed towards the same star describes a cone[Fig. 1.8 (c) and (d)] with vertical angle 2α. The diameter of the circle subtends an angle $2\alpha = 41"$ at the earth. Thus the star appears to move along a circle during a period of one year, when observed from the earth.

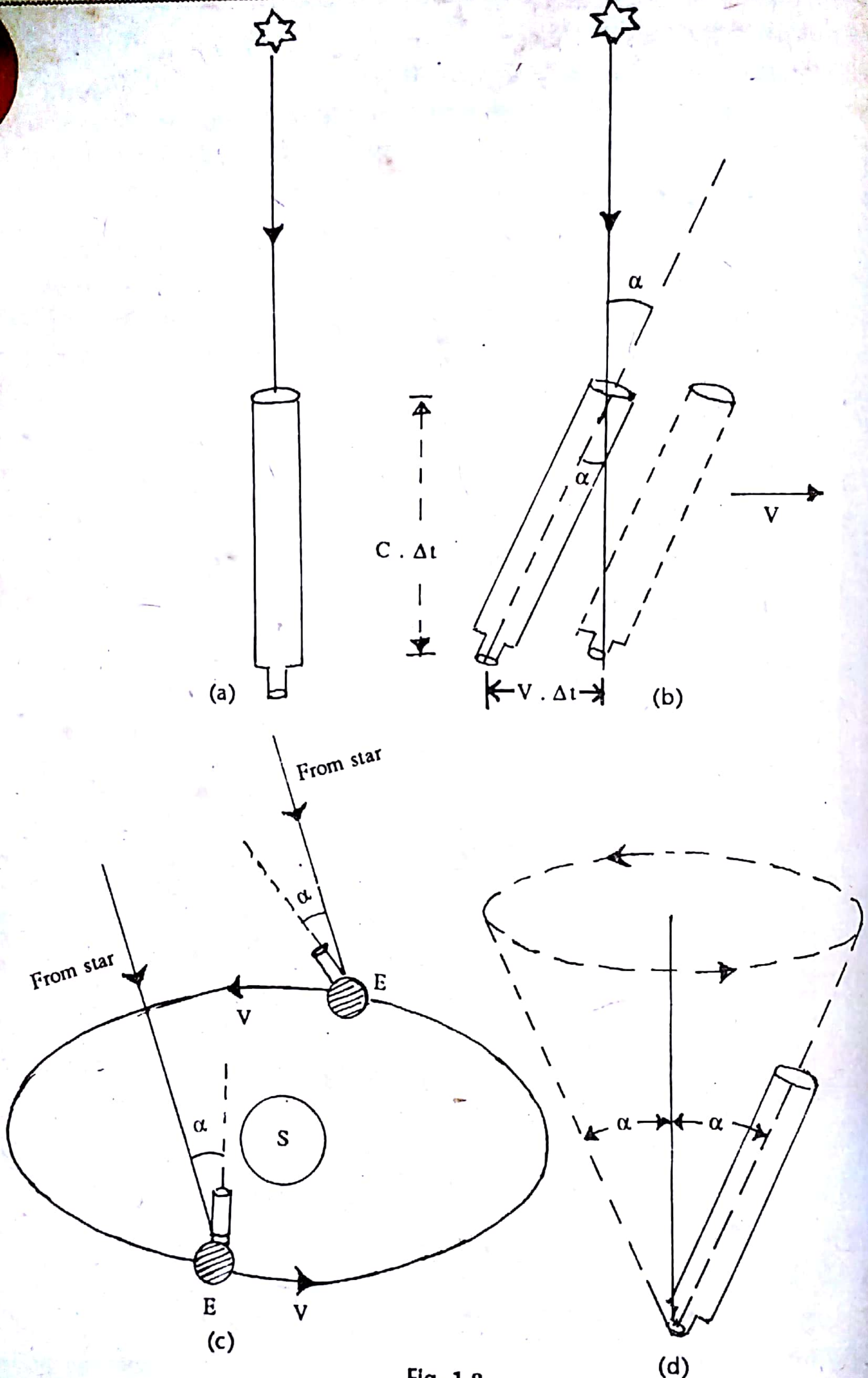

Fig. 1.8

Fig. 1.8 (a) If the earth were at rest relative to the star then the star could have been seen through the telescope by keeping the telescope vertical (b) Since the earth is moving relative to the star with velocity V towards the right the star can be seen through the telescope only when it is kept inclined at an angle $\alpha = 20.5''$. (c) & (d) In one year, the telescope directed towards the same star, describes a cone.

If the ether would have been dragged by the earth then the rays of light also would be moving towards right along with the earth. i.e. the earth would have been stationary relative to the rays. Then the star could have been seen by keeping the telescope vertical. The telescope would not have to be tilted and there would be no aberration of light at all. The star would appear to remain in the same position throughout the year. But the aberration of light has been observed experimentally and it is a well established phenomenon in astronomy. Hence ether is not dragged at all by the earth. If there is ether it remains stationary and the earth moves through ether without disturbing it. Thus the ether-drag hypothesis was rejected.

1.24 LORENTZ - FITZGERALD CONTRACTION HYPOTHESIS :

In order to preserve the concept of unique absolute frame of reference attached to ether and still explain the null result of Michelson-Morley experiment, Lorentz and Fitzgerald put forward a new hypothesis in 1892. It is known as Lorentz-Fitzgerald contraction hypothesis. It is based on the following assumption.

Assumption : The earth and the rays of light travel relative to the absolute frame of reference attached to ether. When a body moves through ether with a velocity V it contracts in the direction of motion, becoming shorter by the factor $\sqrt{1 - \frac{V^2}{c^2}}$. There is no such contraction in the direction perpendicular to the direction of motion.

According to this assumption, the length of a body moving with a velocity V, in the direction of motion is given by

$$l = l_0 \sqrt{1-\frac{V^2}{c^2}}$$

where l_0 is its length when it is stationary relative to the same observer.

$$\text{Since } \sqrt{1-\frac{V^2}{c^2}} < 1, \qquad l < l_0$$

i.e. there is contraction in the length of the moving body in the direction of motion.

Such contraction can not be detected by measuring the length of the moving body using a scale. This is due to the fact that when the scale is kept parallel to the length of the moving body the scale is moving along with the body. Hence the scale also contracts in the direction of motion by the same amount, making the contraction of the body undetectable.

Explanation of the Null Result by the Contraction Hypothesis :

As we have seen in the theory of Mischelson-Morley experiment (§ 1.20),

Total time required by the reflected rays to come back to the plate $= t_1 = \dfrac{2.l}{\sqrt{c^2 - V^2}}$

Total time required by the transmitted rays to come back to the plate $= t_2 = \dfrac{2.lc}{c^2 - V^2}$

Now, according to Lorentz-Fitzgerald contraction hypothesis, the length of the arm of the interferometer in the first case is l (because it is perpendicular to V) whereas in the second case it is $l.\sqrt{1 - \frac{V^2}{c^2}}$ (because it is parallel to V).

$$\therefore t_2 = \frac{2l\sqrt{1 - \frac{V^2}{c^2}} \cdot c}{c^2 - V^2} = \frac{2l\sqrt{c^2 - V^2}}{c^2 - V^2} = \frac{2l}{\sqrt{c^2 - V^2}} = t_1$$

$\therefore$ Time difference between the rays returning to the plate P

$= \Delta t = t_2 - t_1 = 0.$

$\therefore$ Path difference between the rays returning to the plate P

$= \Delta x = 0.$

On rotating the interferometer through 90°, the time difference and hence the path difference would remain zero. Thus there is no shift in the path difference and consequently there will be no shift in the interference pattern. This actually happened in Michelson-Morley experiment.

Thus the contraction hypothesis could explain the null result. But there was no theoretical proof for this type of contraction by the factor $\sqrt{1 - \frac{V^2}{c^2}}$. Moreover, such contraction was never observed in any other experiment (except this experiment!!). Hence the hypothesis was discarded.

1.25 EFFECT OF THE MOTION OF THE SOURCE ON THE VELOCITY OF THE LIGHT :

Towards the end of nineteenth century it was suggested that the velocity of light is $c = 3 \times 10^8$ m/s in empty space relative to the source of light and not relative to the ether. Using this assumption various theories were developed and these theories are called **Emission theories.**

If the source of light is moving with a velocity v_s then according to the above assumption, the velocity of light would be varying from $c - v_s$ to $c + v_s$, depending on the direction of propagation of light.

Now, in Michelson-Morley experiment, the source of light is stationary relative to the apparatus. Hence the velocity of light relative to the apparatus is c in all directions. On rotating the apparatus through 90°, the velocity of light remains the same as before. Hence there cannot be any shift in the interference pattern. There cannot be any effect of earth's motion through ether, on the velocity of light. Thus emission theories could explain the null result.

Emission theories were not consistent with the Maxwell's electromagnetic wave theory of light according to which the velocity of an electromagnetic wave depends upon the electromagnetic property of the transmitting medium and in vacuum it is always equal to $c = 3 \times 10^8$ m/s, whatever may be the motion of the source. According to the wave theory the motion of the source affects the frequency of light (Doppler effect) but not the velocity of light. The emission theories were rejected due to the following experimental evidences.

i) Astronomical observations made by de Sitter on Double Stars (Binary Stars) prove that the velocity of light is not affected by the motion of the stars i.e. by the motion of the sources of light.

ii) Michelson-Morley experiment was performed using sources of light lying outside the earth i.e. using starlight and sunlight. Due to the motion of the earth in space, these sources are moving relative to the apparatus of Michelson - Morley experiment. But no effect of their motion on the velocity of light was detected in the experiments.

1.26 INTERPRETATION OF THE NEGATIVE RESULT IN TERMS OF SPECIAL THEORY OF RELATIVITY :

In 1905, when Einstein formulated the Special theory of Relativity he discarded the idea of an absolute frame of reference attached to ether by saying that there cannot be any unique, universal absolute frame of reference only in which the laws of electrodynamics would be valid. The laws of electrodynamics and all other laws of Physics should be valid and identical in all inertial frames of reference. As regards the validity of these laws all inertial frames are equivalent, there is no preferred or privileged frame of reference in nature.

As we have seen earlier the laws of mechanics are invariant in all inertial frames of reference under Galilean transformations but the laws of electrodynamics are not invariant. Einstein found that the fault lies with the Galilean transformations and not with the laws of electrodynamics. He replaced Galilean transformation equations by a set of new transformation equations called Lorentz transformation equations. He stressed that the Lorentz transformation equations are the correct transformation equations for inertial frames of reference and all laws of mechanics and electrodynamics **both** should be invariant in all inertial frames of reference under Lorentz transformations.

Einstein discarded the idea of ether also. As we have seen in § 1.16 the concept of ether was introduced by Maxwell and Hertz in order to provide a medium through which light and all other electromagnetic waves would travel. In 1905, Einstein himself formulated the photon theory of light according to which light consists of photons and not of waves. Hence there was no need of the medium ether for the propagation of light.

Einstein accepted the experimental facts about the velocity of light that it does not depend upon the motion of the source and the observer. It is one of the basic postulates of his Special theory of Relativity that the velocity of light in empty space is constant. It remains the same in all directions in all inertial frames of reference and does not depend upon the motion of the source and the observer.

Thus according to the Special theory of Relativity, the velocity of light is $c = 3 \times 10^8$ m/s everywhere on the earth. Relative to the apparatus of Michelson -Morley experiment the velocity of light is c in all directions. On rotating the apparatus through 90° the velocity of light remains the same as before. Hence there cannot be any shift in the interference pattern. There cannot be any effect of earth's motion on the velocity of light. Thus the Special theory of Relativity accounts for the negative result.

Note : We have seen in § 1.18 that the velocity of light c in empty space does not remain the same in all inertial frames under Galilean transformations of velocity. We shall prove in the next chapter that the velocity of light, c remains the same in all inertial frames under Lorentz transformations of velocity.

1.27 EFFECT OF THE MOTION OF THE TRANSMITTING MEDIUM ON THE VELOCITY OF LIGHT - FIZEAU'S EXPERIMENT :

Aim of the Experiment :

A ray of light is travelling in a liquid (say, water) having refractive index n. When the liquid is stationary the velocity c_w of the ray relative to an outside observer is given by

$$n = \frac{c}{c_w} \qquad \therefore c_w = \frac{c}{n}$$

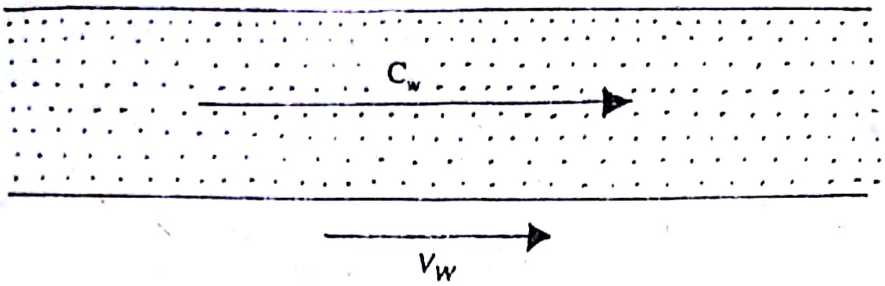

Fig. 1.9 : A ray of light travelling in a flowing liquid

Now, let the liquid be moving with a velocity v_w relative to the same observer. If the liquid drags the ether contained by it along with itself then according to Galilean transformations of velocity, the velocity of the ray relative to the same observer is given by

$$v = c_w \pm v_w = \frac{c}{n} \pm v_w \qquad \text{...(1.17)}$$

Positive sign is taken when the liquid moves in the same direction as the ray of light and negative sign is taken when it travels in the opposite direction.

Using electromagnetic wave theory of light, Fresnel obtained the following formula giving the velocity of the ray travelling in the moving liquid.

$$v = \frac{c}{n} \pm v_w \left(1 - \frac{1}{n^2}\right) \qquad \text{.....(1.18)}$$

The factor $\left(1 - \frac{1}{n^2}\right)$ is called Fresnel's drag coefficient. In order to test which of the above formulae (1.17) and (1.18) is correct, Fizeau in 1851 performed the following experiment.

Apparatus :

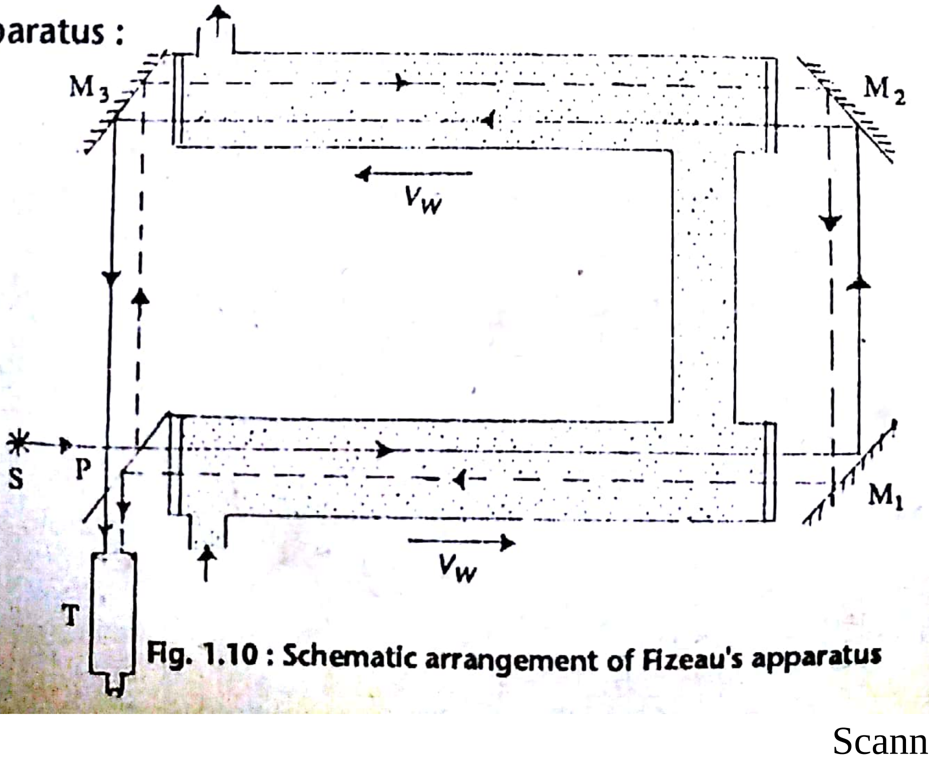

Fig. 1.10 : Schematic arrangement of Fizeau's apparatus

The apparatus used in Fizeau's experiment is shown in the Fig. 1.10. A beam of monochromatic light of wavelength λ from the source S is made incident to a semisilvered glass plate P placed at an angle of 45° with the direction of the incident beam. The beam is partly reflected and partly transmitted in two orthogonal directions.

The transmitted part of the beam travels in anti-clockwise direction and after getting reflected from the mirrors M_1, M_2 and M_3 respectively it comes back to the plate P.

The reflected part of the beam travels in clockwise direction and after getting reflected from the mirrors M_3, M_2 and M_1 respectively it comes back to the plate P.

The rays returning to the plate P are reflected and refracted from the plate and travel towards the telescope T. They interfere with each other along their paths and interference pattern can be seen in the telescope T.

Water flows through the tube as shown in Fig. 1.10. Hence the transmitted part of the beam travels in the direction of the water flow. The reflected part of the beam travels in the direction opposite to that of the water flow.

Method :

Initially the water is kept stationary. Interference pattern is observed in the telescope T.

Then the flow is put on. Water is passed through the tube with high velocity v_w. The transmitted part of the beam travels in the direction of the water flow. According to Fresnel's drag formula (1.18) its velocity is given by

$$v = \frac{c}{n} + v_w\left(1 - \frac{1}{n^2}\right)$$

Let us denote the drag coefficient, $\left(1 - \frac{1}{n^2}\right)$ by d.

$$v = \frac{c}{n} + v_w . d$$

Let l be the length of the path of the beam in each tube. Then the total time required by the transmitted rays to travel in both the tubes is given by.

$$t_1 = \frac{l}{v} + \frac{l}{v} = \frac{2l}{v} = \frac{2l}{\frac{c}{n} + v_w . d}$$

The reflected part of the beam travels in the direction opposite to that of the water flow. Hence its velocity is given by

$$v = \frac{c}{n} - v_w.d$$

Total time required by the reflected rays to travel in both the tubes is given by

$$t_2 = \frac{2l}{v} = \frac{2l}{\frac{c}{n} - v_w.d}$$

Now, t_2 is greater than t_1. Hence the time difference between the reflected and transmitted rays is given by

$$\Delta t = t_2 - t_1$$

$$= \frac{2l}{\frac{c}{n} - v_w.d} - \frac{2l}{\frac{c}{n} + v_w.d}$$

$$= \frac{2l\left(\frac{c}{n} + v_w.d\right) - 2l\left(\frac{c}{n} - v_w.d\right)}{\frac{c^2}{n^2} - v_w^2.d^2}$$

$$= \frac{2l.v_w.d + 2l.v_w.d}{\frac{c^2}{n^2} - v_w^2.d^2} = \frac{4l.v_w.d.n^2}{c^2 - v_w^2.d^2n^2}$$

The term $v_w^2d^2\,n^2$ can be neglected in comparison with c^2.

$$\therefore \Delta t = \frac{4l.v_w.d.n^2}{c^2}$$

$\therefore$ Path difference between the reflected and transmitted rays is given by

$$\Delta x = c.\Delta t = \frac{4l.v_w.d.n^2}{c} \text{ metre} = \frac{4l.v_w.d.n^2}{c\lambda} \text{ wavelengths}$$

When the water is stationary, $v_w = 0$.

$\therefore$ Initial path difference $= \Delta x = 0$.

When the water is flowing with velocity v_w,

Final path difference $= \Delta x = \dfrac{4l\,v_w\,d.n^2}{c\lambda}$ wavelengths.

$\therefore$ Shift in path difference $= \dfrac{4l.v_w.d.n^2}{c\lambda}$ wavelengths

Corresponding shift in the interference pattern

$$= \Delta N = \frac{4l.v_w.d.n^2}{c\lambda} \text{ fringes.}$$

This shift (ΔN) in the interference pattern was actually observed and measured in Fizeau's experiment.

$$\text{Now, } d = \frac{\Delta N.c.\lambda}{4l.v_w.n^2}$$

Substituting the values of ΔN, c, λ, l, v_w and n the drag coefficient (d) was found out.

Result : The experimental value of the drag coefficient (d) coincides with the theoretical value $d = \left(1 - \frac{1}{n^2}\right)$. Thus Fresnel's drag formula was found to be correct in Fizeau's experiment.

Conclusion : If ether would have been dragged by the moving liquid then the velocity of the ray of light would be

$$v = \frac{c}{n} \pm v_w$$

In that case, the value of the drag coefficient would have been $d = 1$. Since it comes out to be $d = \left(1 - \frac{1}{n^2}\right)$ experimentally, we conclude that ether is not at all being dragged by the liquid i.e. the moving liquid does not communicate any of its motion to ether lying inside or outside it.

The velocity of the ray in the moving liquid changes due to the partial drag imparted by the moving liquid as a moving medium.

Problem : In Fizeau's experiment, the values of different measurements were as follows :

$\lambda = 5300$ A°, $v_w = 7$ m/s, $l = 1.5$ m, $\mu = 1.33$.

When water was set into motion a shift of 0.23 fringe was observed. Find the value of drag coefficient. Compare it with its theoretical value.

Given : $\lambda = 5300 \times 10^{-10}$m, $v_w = 7$ m/s,
$l = 1.5$ m, $n = \mu = 1.33$, $\Delta N = 0.23$ fringe, $d = ?$

Solution :

$$d = \frac{\Delta N.c.\lambda}{4l.v_w.n^2} = \frac{0.23 \times 3 \times 10^8 \times 5300 \times 10^{-10}}{4 \times 1.5 \times 7 \times (1.33)^2} = 0.4921$$

Theoretical value of $d = 1 - \frac{1}{n^2}$

$$= 1 - \frac{1}{(1.33)^2}$$

$$= 1 - 0.5652 = 0.4348.$$

QUESTIONS

1. Explain what is an inertial frame and non-inertial frame of reference. Give suitable illustration in each case. (B.U. April 1986, May 1990. Oct. 1992; May 1993)
2. Derive Galilean transformation equations. (B.U. Oct. 1991)
3. What is an invariant quantity? Show that the distance between two points in space is invariant under Galilean transformations.
4. Derive Galilean transformations of velocity and acceleration.
5. What do you mean by invariance of a physical law? Show that the Newton's laws of motion are invariant under Galilean transformations (B.U. Oct. 1991)
6. Show that the Principle of conservation of momentum is invariant under Galilean transformations.
7. Momentum is conserved in collision of two objects as measured by an observer on an uniformly moving train. Show that the momentum is also conserved for an observer on the ground. (B.U.April 1985)
8. Explain the ether hypothesis.
9. Show that the following wave equation of Electrodynamics is not invariant under Galilean transformations.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

10. Explain the idea of Absolute frame of reference.
11. Describe Michelson - Morley experiment. Why was this experiment performed? What was the result obtained? (B.U. May 1981)
12. Describe Michelson - Morley experiment. Derive the necessary formula. Explain the significance of the negative result. (B.U. May 1980, Oct. 1983, April 1985, Oct 1989, April 1991, Oct. 1991, Oct. 1992)
13. Describe Michelson - Morley experiment to determine the speed of earth through ether. Obtain expression for fring shift. Explain the significance of the null result. (B.U. May 1982, April 1983, May 1994)
14. What is ether-drag hypothesis? Why was it dropped?
15. What is Steller aberration? Show that to explain Steller aberration on the basis of Classical theory of light it was necessary to assume that earth move through ether without disturbing it. (B.U. Oct. 1984, April 1992)
16. What is Lorentz - Fitzgerald contraction hypothesis? How does it explain the negative result of Michelson - Morley experiment? Then why was it discarded? (B.U. May 1993, May 1994)
17. Briefly describe the various attempts made to preserve the concept of unique ether frame' and yet explain the null result obtained by Michelson & Morley. (B.U. May 1981, Oct. 1983, Oct 1989.)
18. Explain the Lorentz - Fitzgerald contraction hypothesis and ether drag hypothesis as attempts to preserve the concept of ether. Explain any one observation made in each case as contradiction to the hypothesis. (B.U. April 1987, May 1990, Oct. 1990)
19. How was the null result explained in pre-relativity Physics?
20. How was the null result of Michelson -Morley experiment explained by emission theories? Briefly explain why this theory hold to be rejected. (B.U. April 1983)
21. Describe Michelson-Morley experiment. Explain how the result of the experiment can be interpreted in terms of the special theory of relativity. (B.U. April 1984 & 1989)
22. Describe Michelson-Morley Experiment. Derive the expression giving the fringe-shift expected. How does special theory of relativity explain the unexpected result of Michelson-Morley experiment? (B.U. Oct. 1984)
23. Describe Fizeau's experiment. What are its results and conclusions? (B.U. April 1986, Oct. 1987, April 1988, Oct. 1993)

24. a) State Fresnel's partial drag hypothesis. What is Fresnel's drag coefficient?
    b) Give an account of Fizeau's experiment to determine the Fresnel's drag coefficient. (B.U. April 1985, April 1992)

25. Show that the result of Fizeau's experiment lead to the conclusion that a moving object does not communicate any of its motion to ether inside or outside of it. (B.U. April 1984, April 1986, Oct. 1993)

26. Write notes on the following :
    i) Michelson-Morley experiment. (B.U. May 1989, May 1990, Oct. 1990, Oct. 1993)
    ii) Ether-drag hypothesis and Steller Aberration. (B.U. April 1991, Oct. 1991)
    iii) Lorentz-Fitzgerald contraction hypothesis. (B.U. April 1991,Oct. 1991, April 1992)
    iv) Fizeau's experiment to determine Fresnel's drag coefficient. (B.U. May 1993)

FOOD FOR THOUGHT

1. Absolute space, by its own nature, without relation to anything external, remains always same and immovable.

- Isaac Newton

* * * * *

2. Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external.

- Isaac Newton

* * * * *

3. Absolute motion which is the translation of a body from one absolute place to another absolute place can never be detected; for, translatory motion can be perceived only in the form of motion relative to other material bodies.

- Isaac Newton

* * * * *

4. I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

- Isaac Newton

* * * * *

5. This velocity (of electromagnetic waves) is so nearly that of light that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electro-magnetic laws.

- James Clerk Maxwell

* * * * *

6. But the foundation of electromagnetic theory taught that a particular inertial system must be given preference, namely, that of the luminiferous ether at rest ..... Since the special theory of relativity revealed the physical equivalence of all inertial systems, it proved the untenability of the hypothesis of an ether at rest. It was therefore necessary to renounce the idea that the electromagnetic field is to be regarded as a state of material carrier. The field thus becomes an irreducible element of physical description, irreducible in the same sense as the concept of matter in the theory of Newton.

- Albert Einstein

* * * * *

7. In view of the more recent development of electrodynamics and optics it became more and more evident that classical mechanics affords an insufficient foundation for the physical description of all natural phenomena. *- Albert Einstein*

* * * * *

8. All our attempts to make ether real failed. It revealed neither its mechanical construction nor absolute motion. Nothing remained of all the properties of the ether except that for which it was invented, i.e. its ability to transmit electromagnetic waves. Our attempts to discover the properties of ether led to difficulties and contradictions. After such bad experiences, this is the moment to forget the ether completely and to try never to mention its name. We shall say our space has the physical property of transmitting waves, and so omit the use of a word we have decided to avoid.

- Albert Einstein & Leopold Infeld.
(Evolution of Physics)

* * * * *

2

Einstein's Special Theory of Relativity :

Relativistic Kinematics

"Enough of this, Newton, forgive me; you found the only way which, in your age, was just about possible for a man of highest thought and creative power. The concepts, which you created, are even today still guiding our thinking in Physics, although we now know that they will have to be replaced by others farther removed from the sphere of immediate experience, if we aim at a profounder understanding of relationships".

- *Albert Einstein*

In this chapter we shall formulate the Special theory of Relativity and develop relativistic kinematics. The Special theory of Relativity was put forward by Albert Einstein in 1905. It is founded on two simple postulates which are as follows:

2.1 BASIC POSTULATES OF THE SPECIAL THEORY OF RELATIVITY :

1. **Principle of Relativity :** All the laws of Physics are invariant in all inertial frames of reference i.e. the equations representing the laws of Physics should have the same form in all inertial frames of reference.

2. **Principle of Constancy of the Velocity of Light :** The velocity of light *c* in free space is constant. It remains the same in all directions in all inertial frames of reference.

Explanation :

1. In 1905, ① when Einstein formulated the Special theory of Relativity he discarded the idea of an absolute frame of reference attached to ether by saying that there cannot be any unique, universal absolute frame of reference only in which the laws of electrodynamics would be valid. The laws of electrodynamics and all other laws of Physics should be valid and identical in all frames, at least, in all **inertial frames** of reference. In other words, all the natural laws of Physics should be invariant in all inertial frames of reference. For this purpose

it becomes necessary that the equations representing the laws of Physics should retain their form under co-ordinate transformations from one inertial frame to another. Thus as regards the validity of the laws of Physics, all inertial frames are equivalent. All of them have the same status, there is no preferred or privileged frame in which certain laws would be valid.

The description of a physical event is always relative and varies from one inertial frame to another. Even the values of the physical quantities may change from frame to frame. But the natural laws of Physics should be invariant in all inertial frames of reference. This is the basic philosophy of the Special theory of Relativity. If a physical law is found to be not invariant then it has to be modified and generalized so that it becomes invariant in all inertial frames.

2. As we have discussed in the first chapter, the velocity of light is not invariant under Galilean transformations. i.e. for different observers in different states of motion, the velocity of light comes out to be different. According to Emission theories of Newtonian Relativity the velocity of light emitted from a source depends upon the state of motion of the source.

These theoretical conclusions have been contradicted by the experimental results. Michelson-Morley experiment shows that the velocity of light is not affected by the motion of the observer. The experiments on Double Stars indicate that the velocity of light does not depend upon the state of motion of the source. Einstein accepted all these experimental results and came to the conclusion that the velocity of light c in free space is constant and has the value $c = 3 \times 10^8$ m/s. It remains the same in all directions in all inertial frames of reference and does not depend upon the motion of the source and the observer. This becomes the second postulate of the Special theory of Relativity.

Remark : The second postulate of the Special theory of Relativity can be considered as a corollary of the first postulate. The arguments are as follows:

The propagation of light in the space is governed by the laws and the equations of electrodynamics formulated by Maxwell. According to the first postulate, these laws are invariant in all inertial frames of reference. Hence these laws and the equations remain the same in all inertial frames of reference. This immediately implies that the velocity of light should remain the same in all inertial frames of reference. Thus second postulate is a corollary of the first postulate.

2.2 SEQUENCE OF FURTHER DEVELOPMENT :

We observed in the first chapter that the laws of Mechanics are invariant in all inertial frames of reference under Galilean transformations but the laws of Electrodynamics are not invariant in all inertial frames under Galilean transformations. According to the first postulate of the Special theory of Relativity, the laws of electrodynamics also should be invariant in all inertial frames. Hence either these laws are wrong or Galilean transformations are faulty. Now, the laws of Electrodynamics have been verified experimentally and are well established laws of Physics. Hence Einstein came to the conclusion that the Galilean transformation equations are not the correct transformation equations for inertial frames. He began a search for some new transformation equations.

In 1905, Einstein found that such transformation equations were readily available. H.A. Lorentz had devised them in 1904 while developing his classical theory of electrons. They are now called as Lorentz transformation equations for historical reasons. An important characteristic of these transformation equations is that all laws of electrodynamics are invariant under them. Hence Einstein considered Lorentz transformation equations as the correct transformation equations for inertial frames of reference and they should replace Galilean transformation equations. Einstein stressed that the laws of Mechanics and Electrodynamics both should be invariant in all inertial frames under Lorentz transformations, as required by the first postulate.

This development led to another problem. The laws of Electrodynamics are invariant under Lorentz transformations but the laws of Mechanics are not invariant under Lorentz transformations! Einstein observed that there are discrepancies in our fundamental concepts of space, time, simultaneity and mass which are responsible for the non-invariance of the laws of Mechanics. According to Newton these quantities are absolute and remain the same universally in all inertial frames of reference. Einstein studied these concepts critically and found that in reality they are not absolute but relative i.e. their values are different in different inertial frames of reference. He revised (i.e. modified) these concepts and accordingly generalized the laws of Mechanics so that they would become invariant under Lorentz transformations.

In what follows, we shall present the revision of the concepts of space, time and simultaneity. (We shall study the revision of the concept of mass in the next chapter). Then we shall derive the Lorentz transformation equations in the light of the revised concepts of space, time and simultaneity as Einstein did in 1905.

2.3 COMMUNICATION BETWEEN TWO FRAMES OF REFERENC MOVING RELATIVE TO EACH OTHER :

You must have observed that when lightning occurs in the sky, we see the flash of light first and hear the thundering sound after a small interval of time. When a cracker bursts out at a large distance from you, you must have experienced the same sequence of the flash of light and the exploding sound. You know the reason for this very well. The velocity of light is $c = 3 \times 10^8$m/s whereas that of sound is about 350 m/s only. Hence although light and sound are produced at the same moment the rays of light reach you faster than sound waves because the velocity of light is about 10^6 times higher than that of sound. You must have realised that we can transmit the information through light rays at much faster rate than sound waves.

Light and other electromagnetic waves travel in empty space with the highest velocity available in the universe, viz. $c = 3 \times 10^8$ m/s. Hence in Relativity we shall use the electromagnetic waves (light waves included) to transmit any message, any information, in general, any signal from one inertial frame to another inertial frame. According to the second postulate of the Special theory of Relativity, the velocity of the signal (i.e. the velocity of the electromagnetic waves) remains the same in all inertial frames of reference irrespective of their states of motion.

We get the information of all happenings in our surroundings through the light rays. Are we getting the information instantaneously? No. Never! We are getting the information only after a lapse of time. The reason is that even the light rays do require some time, though very small, to travel from one place to another place because the velocity of light c is finite. Although it is very high **it is not infinite**. If the velocity of light would have been infinite a ray of light would travel all parts of our universe in zero time. This does not happen in reality. We do not have any other way in nature to transmit the signals with a velocity higher than this velocity c. Hence instantaneous transmission of the signal from one place to another place and from one frame to another frame is impossible in nature. **This is the genesis of the relativity of simultaneity, time and space.**

2.4 REVISION OF THE CONCEPTS OF SPACE, TIME AND SIMULTANEITY :

Relativity of Simultaneity :

A and B are two points on the ground near a railway track at a large distance between them. C is the midpoint of AB. [Fig. 2.1]

Two flashes are given out, one from the point A and another from the point B. The observer located on the ground at C finds that the flashes are occuring simultaneously.

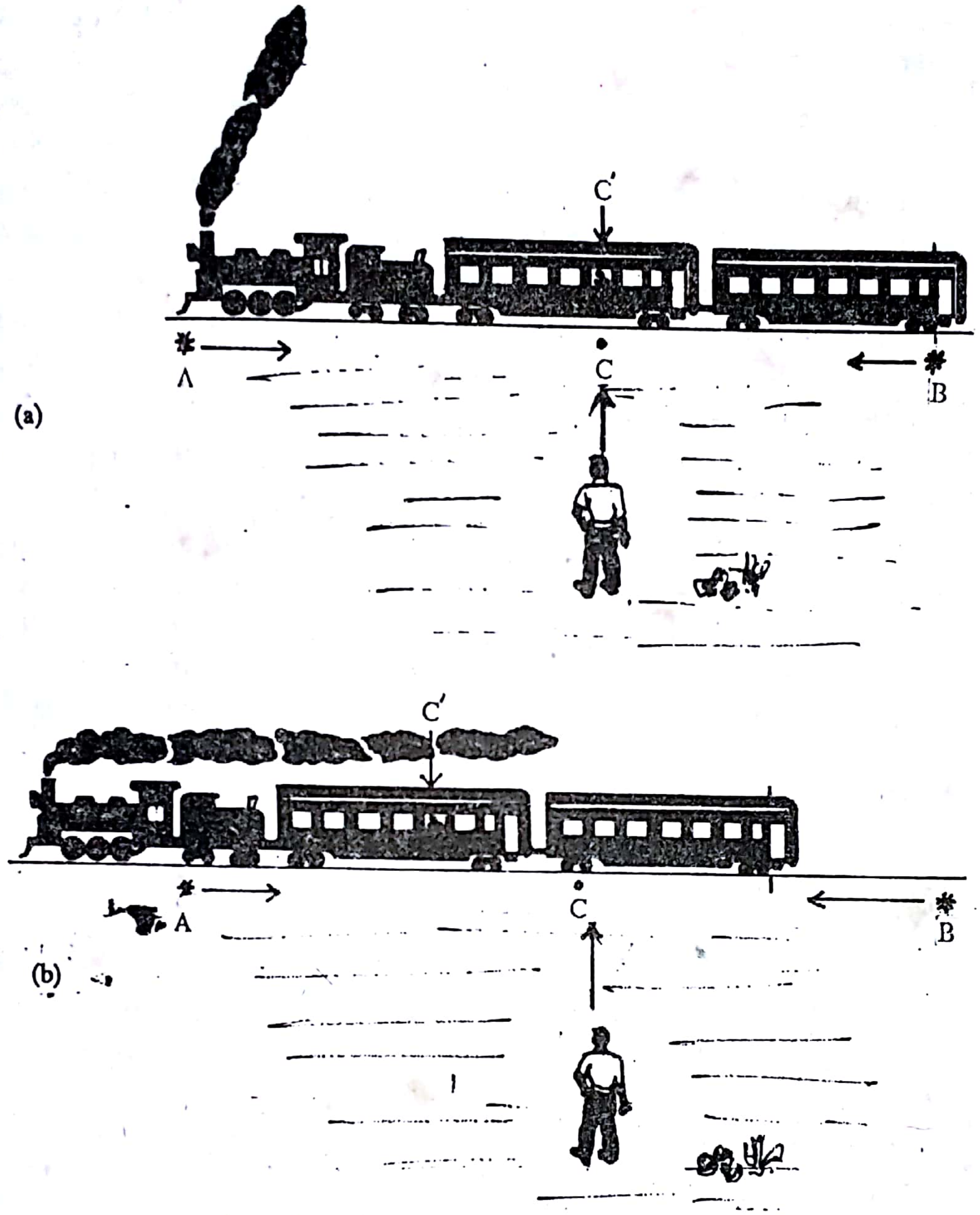

Fig 2.1 (a) The flashes are simultaneous for the observer at C and C'
(b) The flashes are simultaneous for the observer at C but not for the observer at C'

When the train is at rest [Fig. 2.1(a)], the observer in the train at C′, coinciding with C also finds that the flashes are occurring simultaneously.

The train then starts moving. When it is moving with uniform linear velocity V relative to the ground [Fig 2.1 (b)] again the flashes are given out in the same manner so that they are simultaneous for the observer at C. Now, the observer at C' in the moving train observes that the flashes are not occurring simultaneously, the flash from the point A occurs earlier than the flash from the point B. This happens because the light rays emitted from A reach the observer at C' earlier than those from B.

Thus although the events are simultaneous for the observer on the ground, the same events are not simultaneous for the observer in the moving train.

This experiment leads us to conclude that two events occurring at different points, which are simultaneous for one observer, are not simultaneous for other observers in different states of motion i.e. in different frames of reference. Hence simultaneity is a relative concept.

Relativity of Time : Relativity of simultaneity leads to the relativity of time because in all measurements of time we use the idea of simultaneity. For example, when one says : "That train arrives here at 7 o'clock", it means that the pointing of the small hand of clock to 7 and the arrival of the train are simultaneous events.

Now, due to relativity of simultaneity, the events occuring at different points, which are simultaneous in one frame of reference, are not simultaneous for other frames in different states of motion. Hence when the observer on the platform says that the train has arrived exactly at 7 o'clock, for him those two events are simultaneous. But for the observer in some other frame of reference the events will not be simultaneous. He would say that the train reached the station either before or after 7 o'clock but not at 7 o'clock. The time of arrival of the train is 7 o'clock for the observer on the platform whereas it is different from 7 o'clock for the observer in the other frame of reference.

Thus the time of occurrence of an event is different for the frames in different states of motion. Each frame has its own particular local time. Hence time is a relative quantity.

Time can be different for different observers even in one frame. To understand this, let us consider two more illustrations.

Illustration I :

Two persons, one blind (i.e. eyes not working but ears working) and another deaf (i.e. ears not working but eyes working) are sitting side by side. One cracker bursts out at a large distance from them. Each person has some push-button arrangement. When it is pressed the corresponding instant of time is recorded. These persons have been instructed to record the time at which the cracker explodes. They do it dutifully without making any personal error. Now the question is: Will the time recorded by them be the same? You can understand easily that the time recorded by the deaf person (who is getting the information through light rays) is earlier than that recorded by the blind person (who is getting the information through sound waves). Thus the time of occurrence of an event can be different for the observers lying in the same frame of reference!

Now, which is the correct time at which the cracker exploded? You will say that the time recorded by the deaf person is correct. No. The explosion has occurred actually little earlier than this time because the light rays emitted at the time of explosion do require some time to travel from the place of the event to the observer.

Illustration II :

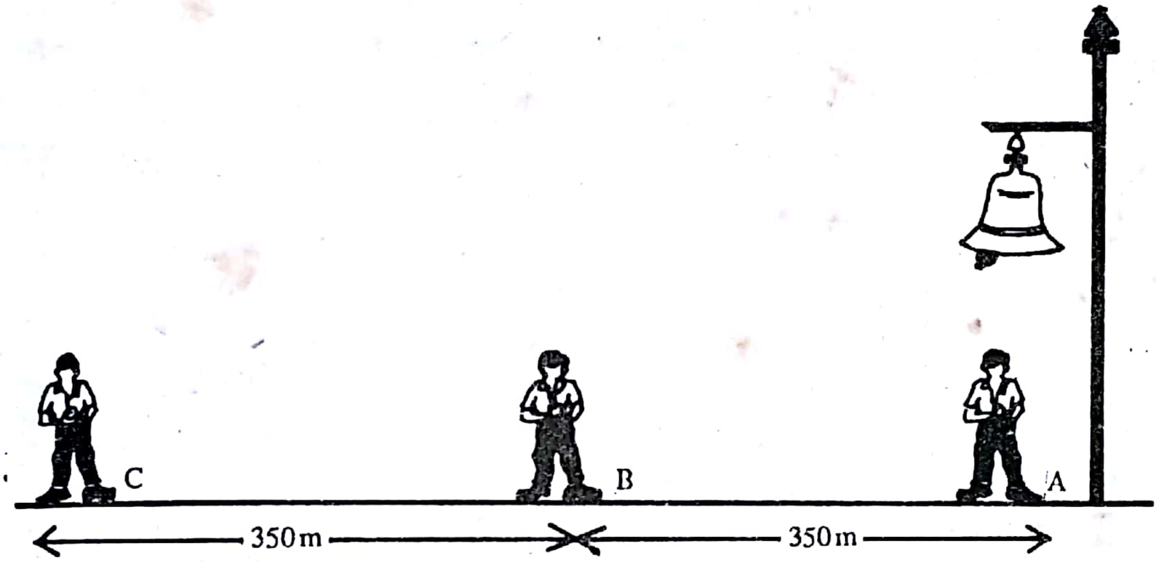

Fig 2.2 Time of ringing of the bell measured by three observers.

There are three observers A, B, and C, each carrying a watch on his wrist. They meet near the bell and synchronize their watches with each other so that they read exactly the same time.

The observer A stands near the bell. (Fig 2.2). The observer B stands at a place lying at a distance 350 m from the bell. (We assume the velocity of sound to be 350 m/s in air) The observer C stands at a place lying at a distance 700 m from the bell. They look into their respective watches and nowhere else.

The bell is given out at one stroke. When these observers hear the bell they record the time instantaneously.

Again they come together near the bell. There is a dispute about the time at which the bell was rung!

A says : "It rang exactly at 9 a.m."
B says : "It rang at 1 sec. past 9 a.m."
C says : "It rang at 2 sec. past 9 a.m."

You can understand why the time of occurrence of a single event is different for these observers. Actually the bell was rung at 9 a.m. The sound waves produced at that time took 1 second to travel through a distance of 350 m and reached the observer B. Hence for the observer B the bell rang at 1 sec. past 9 a.m. The sound waves took 2 seconds to travel through a distance of 700 m and reached the observer C. Therefore for the observer C the bell rang at 2 sec. past 9 a.m.

Whatever is true for sound waves is true for light waves also. If the observer B stands at a distance of 3×10^8 metres i.e. 3×10^5 km from A and looks into the watch of A he would find that the watch of A reads a time which is 1 second behind the time shown by his own watch! The rays of light starting from the watch of A at an instant take 1 second to travel from A to B and reach the observer B after 1 second. Hence B finds that the reading of the watch of A is 1 second behind that of his own watch.

Thus time can be different for the observers located at different places even in one frame of reference.

We get the information of all the happenings in our surroundings through light rays. We are getting the information only after a lapse of small interval of time during which the light rays travel from the place of event to our eyes. How can there be one absolute universal time of occurrence of an event which would be the same for all observers as Newton had thought? Hence we conclude the following :

Time is a relative quantity. It is different for different observers located in different frames of reference. Each frame of reference has its own particular local time.

Relativity of Space : Relativity of simultaneity leads to the relativity of space also because in all measurements of space we use the idea of simultaneity. Let us investigate our process of measurement of length critically.

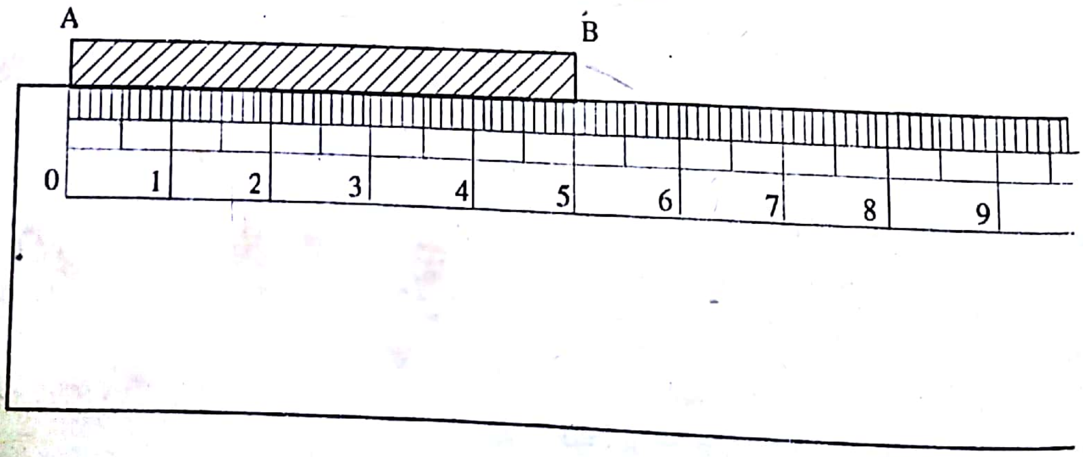

Fig. 2.3 Measurement of a length

When we measure the length of an object AB (Fig. 2.3) we make its end A to coincide with zeroth division of the centimetre scale.

Then the other end B coincides with some division of the scale, say, the division of 5th cm. Hence we say that the length of the object AB is 5cm if when the end A coincides with zeroth division, the end B must coincide with the division of 5th cm. i.e. coincidence of the end A with zeroth division and the coincidence of the end B with the division of 5th cm are simultaneous events. [Do you remember the process of measurement of the length of a moving fish discussed in the first chapter? (see § 1.7, Fig. 1.4). In that case also we have seen that to measure the length of the moving fish its head and tail should be located on the scale simultaneously, at the same instant *t*.]

Now, due to relativity of simultaneity, the events occurring at different points, which are simultaneous in one frame of reference, are not simultaneous for other frames in different states of motion. Hence when the observer in the frame S says that the length of the object AB is 5 cm, for him these two coincidences are simultaneous but for the observer in the frame S', the same coincidences will not be simultaneous and he would say that the length of the object AB is not 5 cm. The length of the object is 5 cm for the observer in the frame S whereas it is different from 5cm for the observer in the frame S'.

Thus the length of an object is different for the frames in different states of motion. Each frame has its own local value of the length of an object. Hence space is a relative quantity.

2.5 LORENTZ TRANSFORMATION EQUATIONS OF SPACE AND TIME COORDINATES :

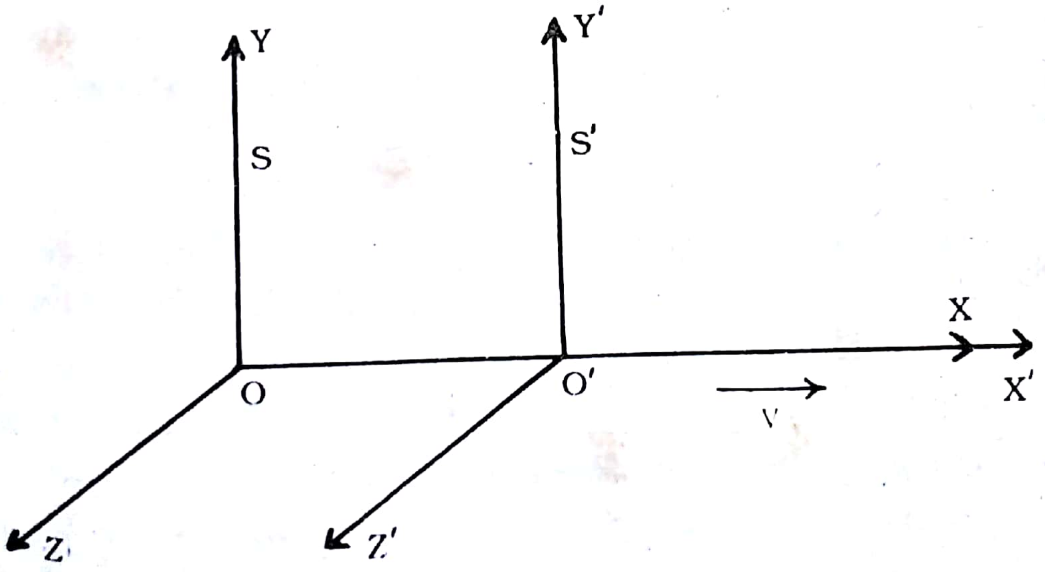

Fig. 2.4 Two inertial frames of reference in relative motion

Let us consider two inertial frames of reference S and S'. The frame S' is moving with uniform linear velocity *V* relative to S along X-axis so that X-axis and X'-axis coincide with each other.

When the origin O' coincides with O momentarily the stop watches are put on in the frame S and S'. Hence at that instant,

$t = 0$ for the frame S and

$t' = 0$ for the frame S'.

At the same instant a flash of light is emitted from a source situated at the coinciding origins. The light rays travel with the same velocity c in both the frames in all directions according to the second postulate of the Special theory of Relativity.

The progress of a ray of light is described relative to the frame S by the equation,

$$c.t = \sqrt{x^2+y^2+z^2}$$

$$\therefore c^2t^2 = x^2+y^2+z^2$$

$$\therefore x^2+y^2+z^2-c^2t^2 = 0 \quad \ldots (2.1)$$

The progress of the same ray is described relative to the frame S' by the equation,

$$c.t' = \sqrt{x'^2+y'^2+z'^2}$$

$$\therefore c^2t'^2 = x'^2+y'^2+z'^2$$

$$\therefore x'^2+y'^2+z'^2-c^2t'^2 = 0 \quad \ldots (2.2)$$

Relative to the observer in the frame S, the origin O' is moving with velocity V along X-axis. The distance covered by the origin O' in time t is given by

$$x = Vt \qquad \therefore x - Vt = 0.$$

Relative to the frame S' the origin O' is characterized by the equation,

$$x' = 0$$

The equations, $x' = 0$ and $x - Vt = 0$ are simultaneous equations characterizing the origin O' with respect to the frame S' and the frame S respectively.

$$\therefore x' = \gamma(x - Vt) \quad \ldots (2.3)$$

where γ is a constant to be determined.

Relative to the observer in the frame S', the origin O is moving with velocity $-V$ along negative X'-axis. The distance covered by the origin O in time t' is given by

$$x' = -Vt' \qquad \therefore x' + Vt' = 0.$$

Relative to the frame S the origin O is characterized by the equation, $x = 0$

The equations $x = 0$ and $x' + Vt' = 0$ are simultaneous equations characterizing the origin O with respect to the frame S and the frame S' respectively.

$$\therefore x = \gamma'(x' + Vt')$$

where γ' is a constant to be determined.

Using equation (2.3), this equation can be written as

$$\begin{aligned} x &= \gamma'[\gamma(x - Vt) + Vt'] \\ &= \gamma\gamma' x - \gamma\gamma' Vt + \gamma' Vt' \\ \therefore \gamma' Vt' &= x - \gamma\gamma' x + \gamma\gamma' Vt. \\ \therefore t' &= \frac{x}{\gamma' V} - \frac{\gamma\gamma' x}{\gamma' V} + \frac{\gamma\gamma' Vt}{\gamma' V}. \\ &= \gamma t + \frac{x}{\gamma' V} - \frac{\gamma x}{V}. \\ &= \gamma\left[t + \frac{x}{\gamma\gamma' V} - \frac{x}{V}\right] \\ &= \gamma\left[t + \frac{x}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right] \end{aligned} \quad \text{...(2.4)}$$

Since there is no relative motion along Y and Z - axes and since X and X' - axes are coinciding,

$$y' = y, \qquad z' = z$$

Thus we have the following transformation equations:

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z.$$

$$t' = \gamma\left[t + \frac{x}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right]$$

We have to find γ and γ'. For this purpose, we shall transform the equation (2.2) into the equation (2.1) using these transformation equations as follows:

The equation (2.2) is

$$\therefore x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

$$\therefore \gamma^2(x - Vt)^2 + y^2 + z^2 - c^2\gamma^2\left[t + \frac{x}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right]^2 = 0$$

$$\therefore \gamma^2(x^2 - 2xVt + V^2t^2) + y^2 + z^2$$
$$- c^2\gamma^2\left[t^2 + \frac{2xt}{V}\left(\frac{1}{\gamma\gamma'} - 1\right) + \frac{x^2}{V^2}\left(\frac{1}{\gamma\gamma'} - 1\right)^2\right] = 0$$

$$\therefore \gamma^2x^2 - 2xVt\gamma^2 + \gamma^2V^2t^2 + y^2 + z^2 - c^2\gamma^2t^2$$

$$- \frac{2xtc^2\gamma^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right) - \frac{c^2\gamma^2x^2}{V^2}\left(\frac{1}{\gamma\gamma'} - 1\right)^2 \quad = \quad 0$$

$$\therefore x^2\left[\gamma^2 - \frac{c^2\gamma^2}{V^2}\left(\frac{1}{\gamma\gamma'} - 1\right)^2\right] + y^2 + z^2$$

$$-2xt\left[V\gamma^2 + \frac{c^2\gamma^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right] + t^2\left[\gamma^2V^2 - c^2\gamma^2\right] \quad = \quad 0$$

Comparing this with equation (2.1) viz.

$$x^2 + y^2 + z^2 - c^2t^2 \quad = \quad 0,$$

$$\gamma^2 - \frac{c^2\gamma^2}{V^2}\left(\frac{1}{\gamma\gamma'} - 1\right)^2 \quad = \quad 1 \qquad \text{...(2.5)}$$

$$V\gamma^2 + \frac{c^2\gamma^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right) \quad = \quad 0 \qquad \text{...(2.6)}$$

$$\gamma^2V^2 - c^2\gamma^2 \quad = \quad -c^2 \qquad \text{...(2.7)}$$

Simplifying equation (2.7) we get

$$\gamma^2(V^2 - c^2) \quad = \quad -c^2$$

$$\therefore \gamma^2 \quad = \quad \frac{-c^2}{V^2 - c^2} \quad = \quad \frac{c^2}{c^2 - V^2} \quad = \quad \frac{1}{1 - \frac{V^2}{c^2}}$$

$$\therefore \gamma \quad = \quad \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Taking equation (2.6)

$$\gamma^2\left[V + \frac{c^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right] \quad = \quad 0$$

As we have seen above, $\gamma \neq 0$.

$$\therefore V + \frac{c^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right) \quad = \quad 0$$

$$\therefore \frac{c^2}{V}\left(\frac{1}{\gamma\gamma'} - 1\right) \quad = \quad -V.$$

$$\therefore \frac{1}{\gamma\gamma'} - 1 \quad = \quad -\frac{V^2}{c^2}$$

$$\therefore \frac{1}{\gamma\gamma'} = 1 - \frac{V^2}{c^2}$$

$$\therefore \gamma\gamma' = \frac{1}{1 - \frac{V^2}{c^2}}$$

$$\therefore \gamma = \frac{1}{\left(1 - \frac{V^2}{c^2}\right)\gamma}$$

$$= \frac{\sqrt{1 - \frac{V^2}{c^2}}}{\left(1 - \frac{V^2}{c^2}\right)}$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Thus we have, $\gamma = \gamma' = \dfrac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$

Hence equation (2.3) becomes,

$$x' = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}(x - Vt) = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Taking equation (2.4), viz,

$$t' = \gamma . \left[t + \frac{x}{V}\left(\frac{1}{\gamma\gamma'} - 1\right)\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[t + \frac{x}{V}\left(\sqrt{1 - \frac{V^2}{c^2}}\sqrt{1 - \frac{V^2}{c^2}} - 1\right)\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[t + \frac{x}{V}\left(1 - \frac{V^2}{c^2} - 1\right)\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[t - \frac{Vx}{c^2}\right]$$

$$= \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Thus we have,

$$\boxed{\begin{aligned} x' &= \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \\ y' &= y, \qquad z' = z. \\ t' &= \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \end{aligned}} \qquad \text{...(2.8)}$$

These are the so-called Lorentz Transformation equations for space and time co-ordinates.

2.6 REDUCTION OF LORENTZ TRANSFORMATION EQUATIONS TO GALILEAN TRANSFORMATION EQUATIONS FOR SMALL VELOCITIES :

Lorentz transformation equations are

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

In Classical Mechanics we deal with the frames of reference (e.g. a railway carriage, an aeroplane etc.) which have velocity V much less than the velocity of light c.

i.e. $V << c$, $\qquad \therefore \frac{V}{c} << 1$.

Hence $\frac{V^2}{c^2}$ becomes negligible in comparison with 1. Even $\frac{Vx}{c^2}$ also becomes negligible in comparison with t. The above equations reduce to

$$x' = x - Vt \qquad y' = y, \qquad z' = z, \qquad t' = t$$

These are nothing but Galilean transformation equations of Newtonian Relativity. Thus Lorentz transformation equations reduce to Galilean transformation equations for small velocities.

This is one of the most beautiful features of the new theory. The Special theory of Relativity does not overthrow the Classical Physics but it is rather a refinement of Classical Physics for velocities comparable to the velocity of light c. (In this connection refer to Food for Thought No. 6, given at the end of this chapter).

2.7 INVERSE LORENTZ TRANSFORMATIONS :

Lorentz transformation equations giving transformations from the frame S' to S are

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad \text{...(2.9)}$$

$$y' = y, \qquad \text{...(2.10)}$$

$$z' = z \qquad \text{...(2.11)}$$

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$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad \text{...(2.12)}$$

We shall obtain the equations giving transformations from the frame S to S' as follows :

i) Considering the equation (2.9),

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$$\text{i.e.} \quad x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore x'\sqrt{1 - \frac{V^2}{c^2}} = x - Vt$$

$$\therefore x = x'\sqrt{1 - \frac{V^2}{c^2}} + Vt. \qquad \text{...(2.13)}$$

Taking the equation (2.12),

$$\text{i.e.} \quad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore t'\sqrt{1 - \frac{V^2}{c^2}} = t - \frac{Vx}{c^2}$$

$$\therefore t = t'\sqrt{1 - \frac{V^2}{c^2}} + \frac{Vx}{c^2}.$$

∴ Equation (2.13) becomes,

$$x = x'\sqrt{1 - \frac{V^2}{c^2}} + Vt'\sqrt{1 - \frac{V^2}{c^2}} + \frac{V^2 x}{c^2}.$$

$$\therefore x\left(1 - \frac{V^2}{c^2}\right) = \sqrt{1 - \frac{V^2}{c^2}}\ (x' + Vt')$$

$$\therefore x = \frac{x' + Vt'}{\sqrt{1 - \frac{V^2}{c^2}}}.$$

ii) Equation (2.10) can be written as

$$y = y'$$

iii) Equation (2.11) can be written as

$$z = z'$$

iv) Considering equation (2.12),

$$\text{i.e. } t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore t'\sqrt{1 - \frac{V^2}{c^2}} = t - \frac{Vx}{c^2}$$

$$\therefore t = t'\sqrt{1 - \frac{V^2}{c^2}} + \frac{Vx}{c^2}$$

By equation (2.13) this equation becomes

$$t = t'\sqrt{1 - \frac{V^2}{c^2}} + \frac{V}{c^2}\left(x'\sqrt{1 - \frac{V^2}{c^2}} + Vt.\right)$$

$$= t'\sqrt{1 - \frac{V^2}{c^2}} + \frac{Vx'}{c^2}\sqrt{1 - \frac{V^2}{c^2}} + \frac{V^2t}{c^2}$$

$$\therefore t\left(1 - \frac{V^2}{c^2}\right) = \sqrt{1 - \frac{V^2}{c^2}}\left(t' + \frac{Vx'}{c^2}\right)$$

$$\therefore t = \frac{t' + \frac{Vx'}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Thus we have

$$x = \frac{x' + Vt'}{\sqrt{1 - \frac{V^2}{c^2}}}. \qquad y = y', \qquad z = z'$$

$$t = \frac{t' + \frac{Vx'}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

These equations give transformations from the frame S to S'. They are called inverse Lorentz transformation equations.

Since the frame S is moving with velocity $-V$ relative to the frame S' these equations can be easily obtained from Lorentz transformation equations by substituting $-V$ for V and interchanging the dashes of space and time co-ordinates.

2.8 SPACE - TIME RELATIONS :

I) Conversion of space into time (or Relativity of simultaneity) :

Let us consider two events occurring at the same instant t but at two different points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively relative to the frame S.

We shall investigate what happens in the frame S' which is moving with uniform linear velocity V relative to S along X-axis.

By Lorentz transformations, $$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

For the first event, $$t'_1 = \frac{t - \frac{Vx_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

For the second event, $$t'_2 = \frac{t - \frac{Vx_2}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

You can see easily that $t'_1 \neq t'_2$ because x_1 and x_2 are different. i.e. the two events are not occurring simultaneously relative to the frame S' although they are simultaneous relative to the frame S. Hence simultaneity is a relative concept. (Do you remember, we have arrived at the same result qualitatively in § 2.4 under the heading "Relativity of simultaneity"?)

The time-interval between the two events as measured in the frame S' is given by

$$t'_2 - t'_1 = \frac{t - \frac{Vx_2}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{t - \frac{Vx_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{-\frac{Vx_2}{c^2} + \frac{Vx_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\frac{V}{c^2}(x_1 - x_2)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

This expression shows that the difference in space positions of the two events in one frame is responsible for the time difference between the two events in the other frame. i.e. space difference is converted into time difference.

If $x_1 = x_2$ then $t'_1 = t'_2$ i.e. if two events occur at the same instant and at the same point relative to the frame S then they will be simultaneous in the frame S' also. But if they occur at different points in the frame S then they will never be simultaneous in the frame S' because the space difference in one frame is converted into time difference in another frame.

ii) Conversion of time into space :

Let us consider two events occurring at the same point (x, y, z) but at two different instants of time t_1 and t_2 respectively relative to the frame S. We shall investigate what happens in the frame S'.

By Lorentz transformations, $x' = \dfrac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$.

For the first event, $x'_1 = \dfrac{x - Vt_1}{\sqrt{1 - \frac{V^2}{c^2}}}$.

For the second event, $x'_2 = \dfrac{x - Vt_2}{\sqrt{1 - \frac{V^2}{c^2}}}$.

You can see easily that $x'_1 \neq x'_2$ because t_1 and t_2 are different. i.e. the two events are not occurring at the same point relative to the frame S' although they occur at the same point relative to the frame S.

The distance between the two events in the direction of X'-axis as measured in the frame S' is given by

$$x'_2 - x'_1 = \frac{x - Vt_2}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{x - Vt_1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{-Vt_2 + Vt_1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{V(t_1 - t_2)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

This expression shows that the time difference between the two events in one frame is responsible for the difference in the space positions of the two events in the other frame. i.e. time difference is converted into the space difference.

If $t_1 = t_2$ then $x'_1 = x'_2$ i.e. if the two events occur at the same point and at the same instant relative to the frame S then they will occur at the same point in the frame S' also. But if they occur at different instants in the frame S then they will never occur at the same point in S' because time difference in one frame is converted into space difference in another frame.

2.9 MINKOWSKI'S FOUR - DIMENSIONAL SPACE :

In Newtonian Relativity the position of a point in space is described by three co-ordinates (x, y, z). Hence the space is considered to be three-dimensional. Moreover space is continuous. Hence we say the space is three-dimensional continuum. In geometry such space is called Euclidean space. Thus Euclidean space is three-dimensional continuum. In Newtonian Relativity, "time is absolute by its very nature flowing uniformly without reference to anything external "i.e. time does not depend upon the space co-ordinates (x, y, z). This is very clearly demonstrated by the Galilean transformation equation for time, viz. $t' = t$. Due to this, we are accustomed to consider time as an independent continuum.

Special theory of Relativity eliminates this special, independent status of time. This is demonstrated by the Lorentz transformation equation for time, namely,

$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

The equation shows that time t' depends upon the space co-ordinate x.

As we have proved in the previous section, the space difference in one frame is converted into time difference in another frame and time difference in one frame is converted into space difference in another frame. Hence in Special theory of Relativity, space and time are intimately and cohesively connected with each other. Moreover they can be converted into each other during co-ordinate transformations from one frame to another frame.

In 1908, H. Minkowski suggested that the world of physical phenomena is four - dimensional and the events occurring in this world are described by four co-ordinates (x, y, z, t). The time co-ordinate t is not independent of the space co-ordinates (x, y, z). It is connected to and equivalent to the space co-ordinates. It plays the same role as the three space co-ordinates. Even the space co-ordinates depend on time co-ordinate. Such space is called Minkowski space.

Thus Minkowski space is four-dimensional space--time continuum. We shall study Minkowski space in more detail in chapter V.

In Newtonian Relativity the world consists of three-dimensional Euclidean space whereas in Special Relativity the world consists of four - dimensional Minkowski space - time.

2.10 INVARIANCE OF THE WAVE EQUATION OF ELECTRO-DYNAMICS UNDER LORENTZ TRANSFORMATIONS :

The electro-magnetic wave equation is

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

where ϕ is scalar potential which is a function of x, y, z, and t.

In § 1.17 of the first chapter, we have seen that this equation is not invariant under Galilean transformations. We shall show that it is invariant under Lorentz transformations in what follows.

i) Using the chain rule of partial derivatives,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\cdot\frac{\partial x'}{\partial x}+\frac{\partial\phi}{\partial y'}\cdot\frac{\partial y'}{\partial x}+\frac{\partial\phi}{\partial z'}\cdot\frac{\partial z'}{\partial x}+\frac{\partial\phi}{\partial t'}\cdot\frac{\partial t'}{\partial x}$$

Now, $$\frac{\partial x'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x-Vt}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial x}{\partial x}-V\frac{\partial t}{\partial x}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}[1-V\times 0] = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\frac{\partial y'}{\partial x} = \frac{\partial y}{\partial x} = 0 \qquad \frac{\partial z'}{\partial x} = \frac{\partial z}{\partial x} = 0,$$

$$\frac{\partial t'}{\partial x} = \frac{\partial}{\partial x}\left[\frac{t-\frac{V.x}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial t}{\partial x}-\frac{V}{c^2}\frac{\partial x}{\partial x}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[0 - \frac{V}{c^2}\right] = \frac{-\frac{V}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial x'}\cdot\frac{1}{\sqrt{1-\frac{V^2}{c^2}}} + \frac{\partial\phi}{\partial y'}\times 0 + \frac{\partial\phi}{\partial z'}\times 0 + \frac{\partial\phi}{\partial t'}\cdot\frac{-\frac{V}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial\phi}{\partial x'} - \frac{V}{c^2}\frac{\partial\phi}{\partial t'}\right]$$

$$\therefore \frac{\partial}{\partial x} = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial}{\partial x'} - \frac{V}{c^2}\frac{\partial}{\partial t'}\right] \quad \text{...(2.14)}$$

ii) $$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x'}\cdot\frac{\partial x'}{\partial y} + \frac{\partial\phi}{\partial y'}\cdot\frac{\partial y'}{\partial y} + \frac{\partial\phi}{\partial z'}\cdot\frac{\partial z'}{\partial y} + \frac{\partial\phi}{\partial t'}\cdot\frac{\partial t'}{\partial y}$$

Now, $$\frac{\partial x'}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial x}{\partial y} - V\frac{\partial t}{\partial y}\right] = 0$$

$$\frac{\partial y'}{\partial y} = \frac{\partial y}{\partial y} = 1, \quad \frac{\partial z'}{\partial y} = \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial t'}{\partial y} = \frac{\partial}{\partial y}\left[\frac{t - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[\frac{\partial t}{\partial y} - \frac{V}{c^2}\frac{\partial x}{\partial y}\right] = 0$$

$$\therefore \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x'}\times 0 + \frac{\partial\phi}{\partial y'}\times 1 + \frac{\partial\phi}{\partial z'}\times 0 + \frac{\partial\phi}{\partial t'}\times 0 = \frac{\partial\phi}{\partial y'}$$

$$\therefore \frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad \text{...(2.15)}$$

iii) Using the same method as in (ii) we can prove,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \qquad \text{...(2.16)}$$

iv) $$\frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x'} \cdot \frac{\partial x'}{\partial t} + \frac{\partial \phi}{\partial y'} \cdot \frac{\partial y'}{\partial t} + \frac{\partial \phi}{\partial z'} \cdot \frac{\partial z'}{\partial t} + \frac{\partial \phi}{\partial t'} \cdot \frac{\partial t'}{\partial t}$$

Now, $$\frac{\partial x'}{\partial t} = \frac{\partial}{\partial t}\left[\frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[\frac{\partial x}{\partial t} - V\frac{\partial t}{\partial t}\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[0 - V\right] = \frac{-V}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\frac{\partial y'}{\partial t} = \frac{\partial y}{\partial t} = 0, \quad \frac{\partial z'}{\partial t} = \frac{\partial z}{\partial t} = 0$$

$$\frac{\partial t'}{\partial t} = \frac{\partial}{\partial t}\left[\frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[\frac{\partial t}{\partial t} - \frac{V}{c^2}\frac{\partial x}{\partial t}\right]$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[1 - \frac{V}{c^2} \times 0\right] = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x'} \cdot \frac{-V}{\sqrt{1 - \frac{V^2}{c^2}}} + \frac{\partial \phi}{\partial y'} \times 0 + \frac{\partial \phi}{\partial z'} \times 0 + \frac{\partial \phi}{\partial t'} \cdot \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[\frac{\partial \phi}{\partial t'} - V\frac{\partial \phi}{\partial x'}\right]$$

$$\therefore \frac{\partial}{\partial t} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left[\frac{\partial}{\partial t'} - V\frac{\partial}{\partial x'}\right] \qquad \text{...(2.17)}$$

The electromagnetic wave equation in the frame S is as follows :

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0 \qquad ...(2.18)$$

$$\therefore \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\phi = 0$$

By equations (2.14), (2.15), (2.16) and (2.17) this equation becomes,

$$\left[\frac{1}{1-\frac{V^2}{c^2}}\left(\frac{\partial}{\partial x'}-\frac{V}{c^2}\frac{\partial}{\partial t'}\right)^2+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\right.$$

$$\left.\frac{1}{c^2}\frac{1}{1-\frac{V^2}{c^2}}\left(\frac{\partial}{\partial t'}-V\frac{\partial}{\partial x'}\right)^2\right]\phi' = 0$$

$$\therefore \left[\frac{c^2}{c^2-V^2}\left(\frac{\partial^2}{\partial x'^2}-\frac{2V}{c^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{V^2}{c^4}\frac{\partial^2}{\partial t'^2}\right)+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right.$$

$$\left.-\frac{1}{c^2}\frac{c^2}{c^2-V^2}\left(\frac{\partial^2}{\partial t'^2}-2V\frac{\partial^2}{\partial t'\partial x'}+V^2\frac{\partial^2}{\partial x'^2}\right)\right]\phi = 0$$

$$\therefore \left[\frac{c^2}{c^2-V^2}\frac{\partial^2}{\partial x'^2}-\frac{2V}{c^2-V^2}\frac{\partial^2}{\partial x'\partial t'}+\frac{V^2}{c^2(c^2-V^2)}\frac{\partial^2}{\partial t'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right.$$

$$\left.-\frac{1}{c^2-V^2}\frac{\partial^2}{\partial t'^2}+\frac{2V}{c^2-V^2}\frac{\partial^2}{\partial t'\partial x'}-\frac{V^2}{c^2-V^2}\frac{\partial^2}{\partial x'^2}\right]\phi = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x'^2}\left(\frac{c^2}{c^2-V^2}-\frac{V^2}{c^2-V^2}\right)+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}\right.$$

$$\left.+\frac{\partial^2}{\partial t'^2}\left\{\frac{V^2}{c^2(c^2-V^2)}-\frac{1}{c^2-V^2}\right\}\right]\phi = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x'^2}\left(\frac{c^2-V^2}{c^2-V^2}\right)+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}+\frac{\partial^2}{\partial t'^2}\left\{\frac{V^2-c^2}{c^2(c^2-V^2)}\right\}\right]\phi = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{\partial^2}{\partial t'^2}\left\{\frac{c^2-V^2}{c^2(c^2-V^2)}\right\}\right]\phi = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\phi = 0$$

$$\therefore \frac{\partial^2\phi}{\partial x'^2}+\frac{\partial^2\phi}{\partial y'^2}+\frac{\partial^2\phi}{\partial z'^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = 0 \qquad ...(2.19)$$

This is the wave equation in the frame S'. It has the same form as equation (2.18) referring to the frame S. Hence the electromagnetic wave equation is invariant under Lorentz transformations.

Note : This is the reason why Einstein considered Lorentz transformation equations as the correct transformation equations for inertial frames of reference and stressed that they should replace Galilean transformation equations. Moreover they do reduce to Galilean transformation equations for small velocities.

2.11 COVARIANT FORMULATIONS :

According to the first postulate of the Special theory of Relativity all the laws of Physics should be invariant in all inertial frames of reference. i.e. when we transform the equation representing a law of Physics from one inertial frame to another inertial frame using Lorentz transformation equations, the form of the equation should remain the same. Only the x, y, z, t co-ordinates of the frame S must change to x', y', z', t' of the frame S' respectively. In the previous section we have seen how this happens for the electromagnetic wave equation.

An equation which retains its form under co-ordinate transformations is said to have covariant form.

In constructing any equation it is always required that the terms of the equation should be all scalars or all vectors. The Special theory of Relativity imposes another condition : The equation should have covariant form under Lorentz transformations. This in turn provides a test for the correctness of the equation representing a law of Physics. If an equation is not having covariant form then it has to be modified and generalized. According to Einstein, "This is a definite mathematical condition that the Special theory of Relativity demands of a natural law and by virtue of this the theory becomes a valuable heuristic aid in the search for the general laws of nature."

In reality, this condition did lead to many new developments in theoretical Physics. We will consider one such development in what follows :

Schrodinger's time-dependent wave equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + U\Psi$$

i.e.
$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2}\right] + U\Psi$$

contains first derivative in time and second derivative in space co-ordinates. There is no symmetry in space and time co-ordinates. Hence if we carry out the Lorentz transformations the equation will not retain its form. Thus the equation is not having covariant form and needs to be modified and generalized. P.A.M. Dirac took up this task and in 1928 developed a new equation having covariant form under Lorentz transformations. It is called Dirac Equation. This equation established theoretically the existence of the intrinsic spin of electrons and the existence of antiparticles also. Schrodinger's time-dependent equation has no capacity to achieve these results!

Problem : Show that the quantity $x^2 + y^2 + z^2 - c^2t^2$ is invariant under Lorentz transformations of space-time co-ordinates.

Solution : Lorentz transformations of space and time co-ordinates are

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \qquad y' = y, \qquad z' = z, \qquad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

In the frame S', the quantity has the form,

$$x'^2 + y'^2 + z'^2 - c^2t'^2$$

$$= \frac{(x - Vt)^2}{1 - \frac{V^2}{c^2}} + y^2 + z^2 - \frac{c^2\left(t - \frac{Vx}{c^2}\right)^2}{1 - \frac{V^2}{c^2}}$$

$$= \frac{(x - Vt)^2 - c^2\left(t - \frac{Vx}{c^2}\right)^2}{1 - \frac{V^2}{c^2}} + y^2 + z^2$$

$$= \frac{(x^2 - 2xVt + V^2t^2) - c^2\left(t^2 - \frac{2xVt}{c^2} + \frac{V^2x^2}{c^4}\right)}{1 - \frac{V^2}{c^2}} + y^2 + z^2$$

$$= \frac{x^2 - 2xVt + V^2t^2 - c^2t^2 + 2xVt - \frac{V^2x^2}{c^2}}{1 - \frac{V^2}{c^2}} + y^2 + z^2$$

$$= \frac{x^2\left(1 - \frac{V^2}{c^2}\right) + t^2(V^2 - c^2)}{1 - \frac{V^2}{c^2}} + y^2 + z^2$$

$$= x^2 + \frac{t^2(V^2 - c^2)}{c^2 - V^2}c^2 + y^2 + z^2$$

$$= x^2 - \frac{c^2t^2(c^2 - V^2)}{c^2 - V^2} + y^2 + z^2$$

$$= x^2+y^2+z^2-c^2t^2$$

Thus the value of the quantity $x^2+y^2+z^2-c^2t^2$ remains the same in both the frames S and S'. Hence the quantity remains invariant under Lorentz transformations of space-time co-ordinates.

Problem : There is a plane in the frame S on which the clocks in S and S' always agree. prove that this plane moves with velocity $\frac{c^2}{V}\left[1-\sqrt{1-\frac{V^2}{c^2}}\right]$ relative to S. Show that this velocity is less than V.

Solution : Let a point (x, y, z, t) be lying on the plane.

By definition of the plane, $t = t'$.

By Lorentz transformations,

$$t' = \frac{t-\frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore t = \frac{t-\frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore t\sqrt{1-\frac{V^2}{c^2}} = t-\frac{Vx}{c^2}$$

$$\therefore \frac{Vx}{c^2} = t\left[1-\sqrt{1-\frac{V^2}{c^2}}\right]$$

$$\therefore \frac{x}{t} = \frac{c^2}{V}\left[1-\sqrt{1-\frac{V^2}{c^2}}\right]$$

$\therefore$ Velocity of the plane relative to the frame S

$$= \frac{x}{t} = \frac{c^2}{V}\left[1-\sqrt{1-\frac{V^2}{c^2}}\right]$$

Let us assume that this velocity is not less than V. Hence it must be grater than or equal to V

$$\text{i.e. } \frac{c^2}{V}\left[1-\sqrt{1-\frac{V^2}{c^2}}\right] \geq V$$

$$\therefore 1-\sqrt{1-\frac{V^2}{c^2}} \geq \frac{V^2}{c^2}$$

$$\therefore 1-\frac{V^2}{c^2} \geq \sqrt{1-\frac{V^2}{c^2}}$$

This is absurd due to the following reason. Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1-\frac{V^2}{c^2}}$ is always less than 1. Hence its square, $1-\frac{V^2}{c^2}$ should be less than $\sqrt{1-\frac{V^2}{c^2}}$

$\therefore$ The velocity of the plane must be less than V.

Problem : Two events A and B occur at the same point in the frame S but B later than A. Prove that the sequence remains the same in all frames.

Solution :

Let the events A and B occur at the same point (x, y, z) in the frame S but at the instants t_1 and t_2 respectively. Since the event B occurs later than A, $t_2 > t_1$.

We shall investigate what happens in the frame S' which is moving with uniform linear velocity V relative to S along X - axis.

By Lorentz transformations, $t' = \dfrac{t - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$

For the event A, $t_1' = \dfrac{t_1 - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$

For the event B, $t_2' = \dfrac{t_2 - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$

$$\therefore t_2' - t_1' = \frac{t_2 - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{t_1 - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{t_2 - \frac{Vx}{c^2} - t_1 + \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{t_2 - t_1}{\sqrt{1-\frac{V^2}{c^2}}}$$

Since $t_2 > t_1$, $\quad t_2 - t_1 > 0$.

$\therefore t'_2 - t'_1 > 0 \qquad \therefore t'_2 > t'_1$

i.e. the event B occurs later than A in the frame S' also. Thus for al frames, B occurs later than A and the sequence of the events is conserved.

Note : The sequence of events is conserved for all frames because the two events are occurring at the same point in S. If they occur at different points then,

$$t'_1 = \frac{t_1 - \frac{Vx_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad t'_2 = \frac{t_2 - \frac{Vx_2}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore t_2' - t_1' = \frac{t_2 - \frac{Vx_2}{c^2} - t_1 + \frac{Vx_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{(t_2 - t_1) - \frac{V}{c^2}(x_2 - x_1)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

If the distance between the two events, $(x_2 - x_1)$ is very large, the velocity V of the frame S' also is very large and time interval ($t_2 - t_1$) between the events is very small then it can happen that

$$t_2 - t_1 < \frac{V}{c^2}(x_2 - x_1)$$

Then $t'_2 - t'_1 < 0$ i.e. $t'_2 < t'_1$ or $t'_1 > t'_2$ i.e. the event A occurs later than B. i.e. the sequence of events may not be preserved! Thus even the sequence of occurance of events may change from one frame to another frame. This is the reason for naming the theory as "Theory of Relativity". (It is necessary here that these events must not be connected with each other as cause and effect).

2.12 LORENTZ - FITZGERALD LENGTH CONTRACTION :

Let us consider an uniform rectangular bar lying along X'-axis in the frame S'. (Fig. 2.5). The frame S' is moving with uniform linear velocity V relative to the frame S along X-axis. Hence the bar is stationary relative to the frame S' whereas it is moving with velocity V along X-axis relative to the frame S.

Let x'_1 and x'_2 be the X- co-ordinates of the end points of the bar located along its length relative to the frame S'. Let x_1 and x_2 be the corresponding X- co-ordinates relative to the frame S.

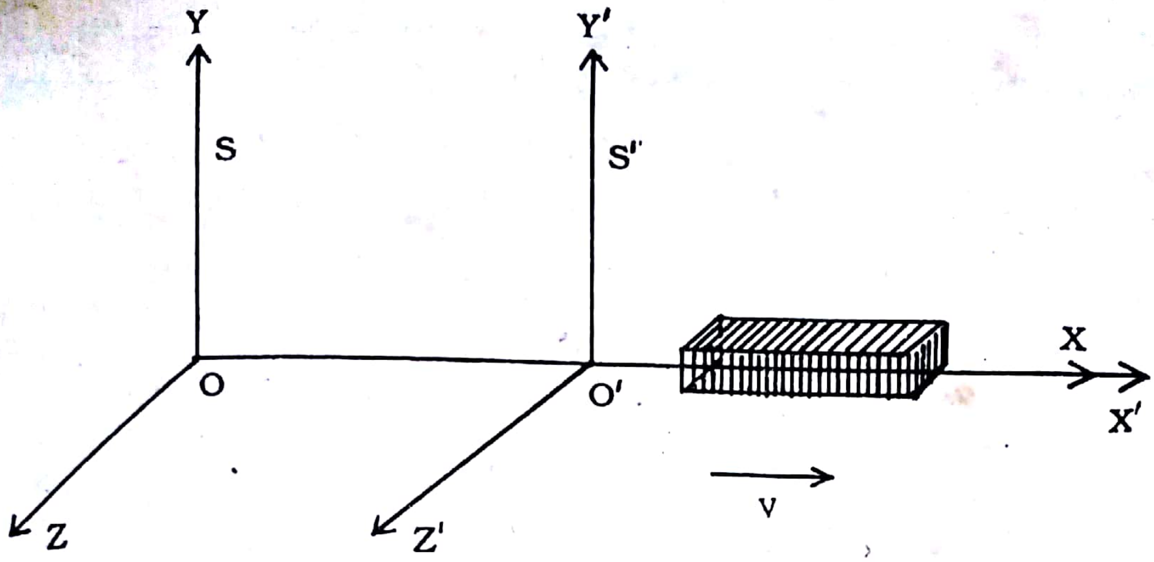


Fig 2.5 Rectangular bar is at rest in frame S', but moving relative to the frame S

By Lorentz transformations,

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore x_1' = \frac{x_1 - Vt}{\sqrt{1 - \frac{V^2}{c^2}}},$$

$$x_2' = \frac{x_2 - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

While measuring the length of an object the end points of the object should be located on the axis simultaneously i.e. at the same instant *t*. Hence we have taken the same *t* in the above two equations. [In this connection, recollect the process of measurement of the length of a moving fish discussed in the first chapter (see §1.7, Fig. 1.4)]

$$\therefore x_2' - x_1' = \frac{x_2 - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{x_1 - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{x_2 - Vt - x_1 + Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{x_2 - x_1}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad \text{...(2.20)}$$

Now, $(x'_2 - x'_1)$ is the length of the bar measured in the frame S' in which it is stationary. This length is called **Proper Length** of the bar and is denoted by $(\Delta x)_0$

$(x_2 - x_1)$ is the length of the same bar measured in the frame S relative to which the bar is moving with velocity V. This length is called **Observed Length** or non-proper length and is denoted by Δx.

∴ Equation (2.20) can be written as

$$(\Delta x)_0 = \frac{\Delta x}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore \Delta x = \sqrt{1 - \frac{V^2}{c^2}}\,(\Delta x)_0$$

Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1 - \frac{V^2}{c^2}}$ is always less than 1.

$$\therefore \Delta x < (\Delta x)_0$$

i.e. the observed length of the moving bar is less than its proper length.

Thus while moving with velocity V the bar appears to be contracted in the direction of motion by the factor $\sqrt{1 - \frac{V^2}{c^2}}$ ...(a)

Let y'_1 and y'_2 be the Y- co-ordinates of the end points of the bar located along its height relative to the frame S'. Let y_1 and y_2 be the corresponding Y - co-ordinates relative to the frame S.

By Lorentz transformations, $y' = y$.

$$\therefore y'_1 = y_1, \qquad y'_2 = y_2.$$

$$\therefore y'_2 - y'_1 = y_2 - y_1.$$

$$\therefore (\Delta y)_0 = \Delta y \qquad \text{i.e.} \quad \Delta y = (\Delta y)_0$$

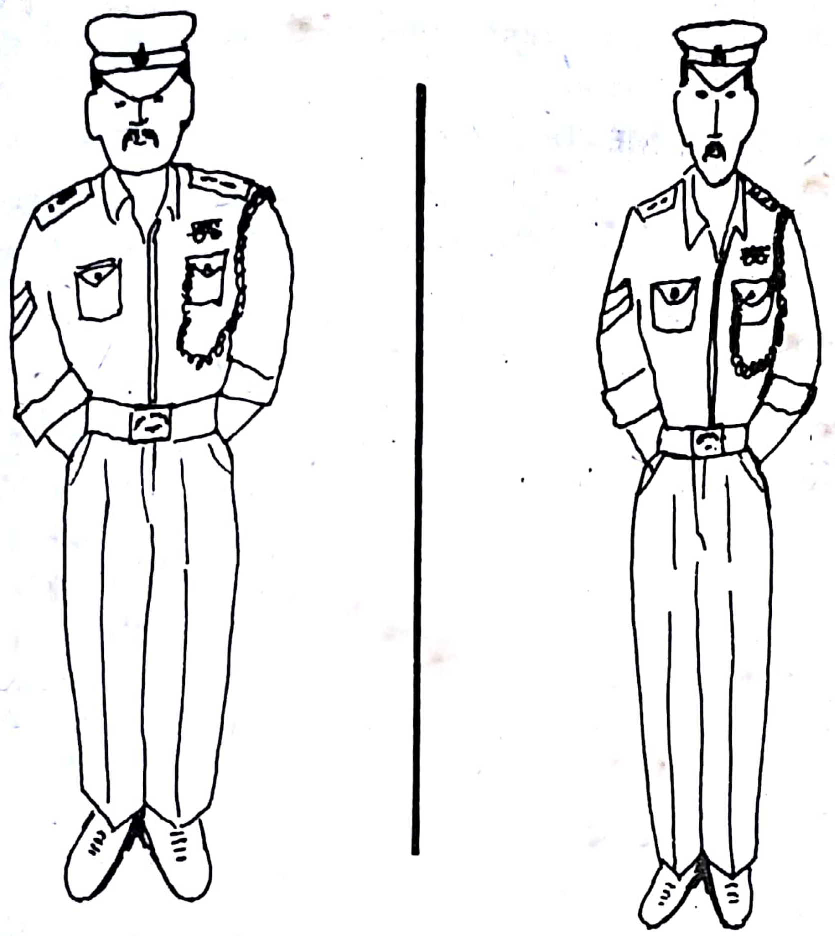
Similarly for breadth,

$$\Delta z = (\Delta z)_0$$

These two results show that no contraction appears along the height and the breadth of the bar, i.e. in the direction perpendicular to the direction of motion. ...(b)

Thus we conclude from (a) and (b) that while moving with velocity V the bar appears to be contracted in the direction of motion by the factor $\sqrt{1 - \frac{V^2}{c^2}}$ but there is no such contraction in the direction perpendicular to the direction of motion. This phenomenon is called Lorentz-Fitzgerald length-contraction.

The phenomenon is illustrated in Fig 2.6.



(a) (b)

Fig. 2.6 (a) Railway policeman, as observed from the ground.
(b) The same Railway policeman, as observed from a train moving with a velocity 0.8 c.

Remarks :

i) As we have seen here the Lorentz-Fitzgerald length contraction is a natural consequence of the Lorentz transformations of the Special theory of Relativity. As we have seen in § 1.24 this contraction hypothesis could explain the negative result of Michelson-Morley experiment. Hence the Special theory of Relativity accounts for the negative result, indirectly.

ii) The following equation represents the law of transformation of length.

$$\boxed{\therefore \Delta x = (\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}}}$$

It shows that for an observer the length of a moving object varies with respect to its velocity V relative to the same observer. Thus the space is not absolute but relative.

iii) Due to length contraction, all observed lengths of a body are less than its proper length. Hence the proper length of the body is its maximum length.

2.13 EINSTEIN'S TIME - DILATION :

In order to understand the phenomenon of dilation (i.e. expansion) of time, first revealed by Einstein, we shall study the following illustration :

Illustration :

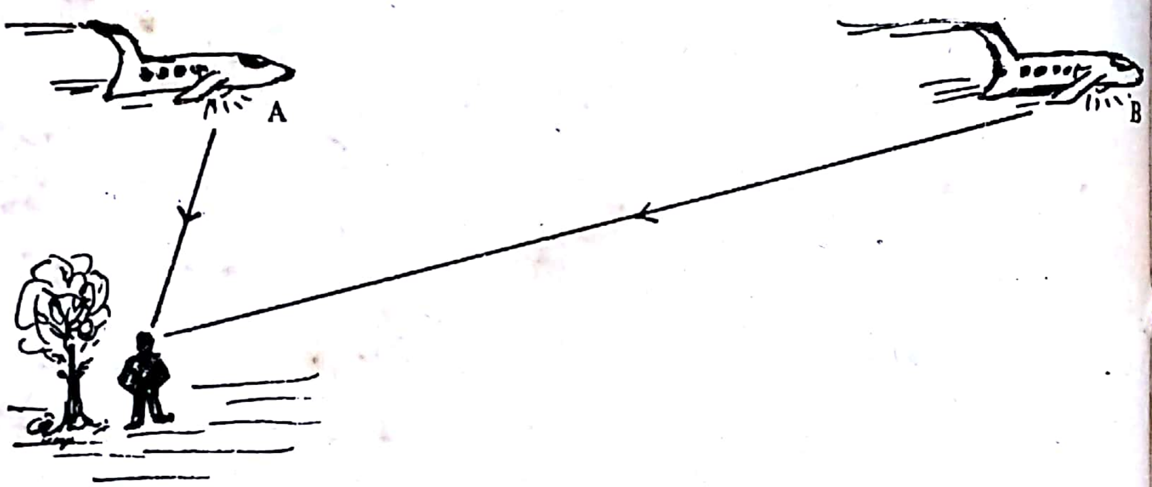


Fig. 2.7 Measurement of time-interval between two flashes of light emitted by an aeroplane

At night, one aeroplane is travelling in the dark sky along a straight line with uniform linear velocity V (Fig. 2.7) We assume that this velocity V is comparable to the velocity of light c. As usual, it emits the flashes of light at certain interval of time.

There are two observers who want to measure the time-interval between two successive flashes of light emitted from the aeroplane. One of them is travelling by the same aeroplane. The other observer is standing on the ground. Both the observers have exactly the same type of stop-watches.

When the aeroplane is at A it emits a flash of light. When the observers receive the rays of light they start their stop-watches immediately.

The aeroplane moves forward and emits the next flash of light from the position B. When the observers receive the rays of light they stop their stop-watches.

What is the result of the measurements? The observer in the aeroplane finds that the time - interval between the two flashes is 1 second. The observer on the ground finds that the time-interval between the same two flashes is greater than 1 second! How can the time - interval between the same two events be different for these observers? Why does the time-interval appear to be dilated for the ground observer? The reason is as follows.

For the observer in the aeroplane the two flashes were emitted from the same point on the aeroplane. Hence the time-interval between the two flashes is "really" 1 second (i.e. it is proper time-interval).

For the observer on the ground one flash was emitted from the point A and the second flash was emitted from a different point B. The place of events was moving with velocity *V*. The ray of light emitted from B has to travel a greater distance than the ray emitted from A. Since the velocity of light is finite it does require some time to travel the extra distance. Hence the time-interval between the arrivals of the rays at the ground observer is greater than 1 second.

Thus if the place of events is moving with high velocity comparable to the velocity of light the observed time-interval between two events appears to be expanded as compared to the proper time-interval. This phenomenon is called Einstein's time-dilation.

Derivation of the formula :

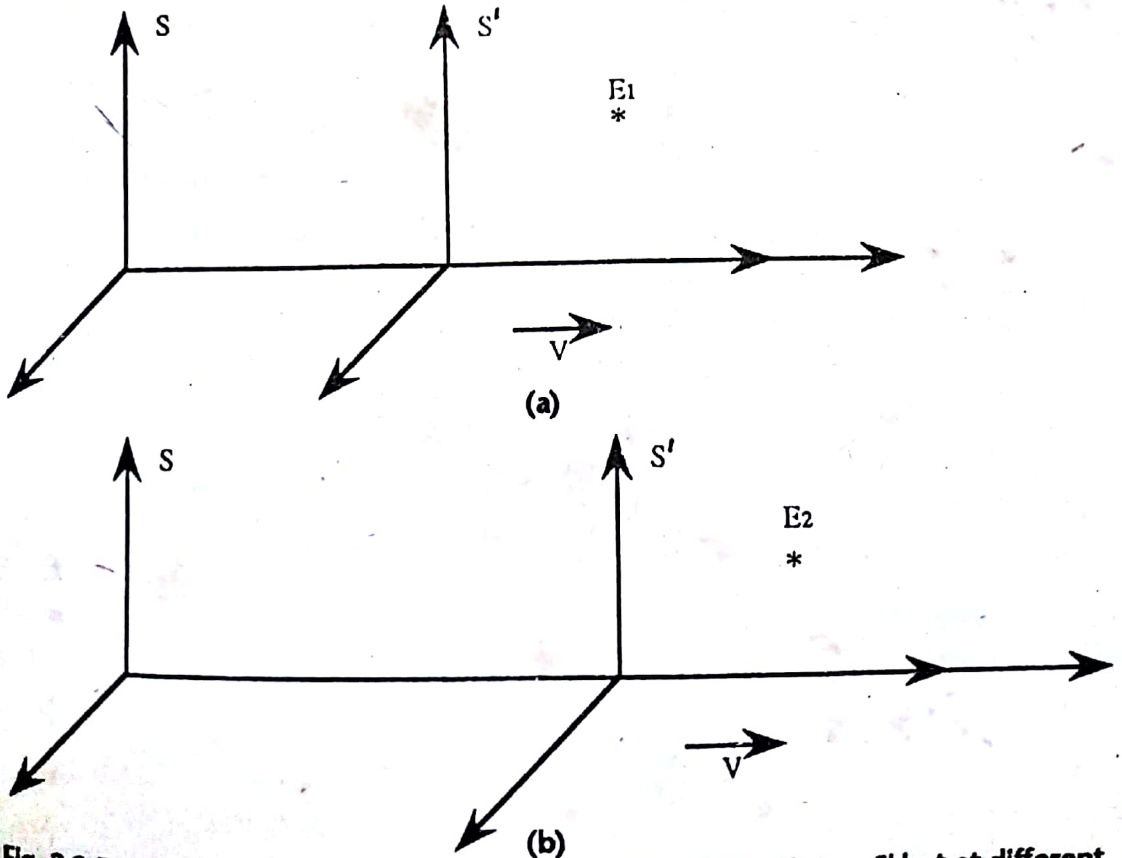


Fig. 2.8 Events E_1 and E_2 occurring at the same place in the frame S' but at different places in the frame S

An event E_1 occurs in space. It is described in the frame S' by the co-ordinates (x'_1, y'_1, z'_1, t'_1) and in the frame S by the co-ordinates (x_1, y_1, z_1, t_1). [See Fig. 2.8.(a)]

Another event E_2 occurs at the same point relative to the frame S' but at a later instant t'_2. Hence it is described in the frame S' by the co-ordinates (x'_1, y'_1, z'_1, t'_2). Since the frame S' is moving with velocity V relative to the frame S this event occurs at a different point in the frame S and it is described as (x_2, y_2, z_2, t_2). [See Fig. 2.8 (b)].

Thus the place of events is stationary relative to the frame S' whereas it is moving with velocity V relative to the frame S along X-axis.

By Lorentz transformations, $$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Taking inverse transformations, $$t = \frac{t' + \frac{Vx'}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

For the first event E_1, $$t_1 = \frac{t'_1 + \frac{Vx'_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

For the second event E_2, $$t_2 = \frac{t'_2 + \frac{Vx'_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore t_2 - t_1 = \frac{t'_2 + \frac{Vx'_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{t'_1 + \frac{Vx'_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{t'_2 + \frac{Vx'_1}{c^2} - t'_1 - \frac{Vx'_1}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{t'_2 - t'_1}{\sqrt{1 - \frac{V^2}{c^2}}} \qquad ...(2.21)$$

Now, $(t'_2 - t'_1)$ is the time-interval between the two events as measured by the observer in the frame S' in which the event occur at the same place i.e. the place of events is stationary. This time-interval is called **proper time-interval** between the two events and is denoted by $(\Delta t)_0$

$(t_2 - t_1)$ is the time - interval between the same two events as measured by the observer in the frame S relative to which the place of events is moving with velocity V. This time-interval is called **observed time-interval** or non - proper time interval and is denoted by Δt.

$\therefore$ Equation (2.21) can be written as

$$\Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1 - \frac{V^2}{c^2}}$ is always less than 1

$$\therefore \Delta t > (\Delta t)_0$$

Thus if the place of events is moving the observed time - interval between two events appears to be expanded (i.e. dilated) as compared to the proper time-interval. The formula representing Einstein's time-dilation is

$$\therefore \Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Remarks :

i) The following equation represents the law of transformation of time - intervals.

$$\boxed{\Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}}$$

It shows that for an observer the time-interval between two events varies with respect to the velocity V at which the place of events is moving relative to the same observer. Thus the time - interval is not absolute but relative.

ii) Due to time-dilation all observed time-intervals between two events are greater than the proper time interval. Hence the proper time-interval is the shortest time-interval between the two events.

iii) Due to Einstein's time-dilation, one second of a moving clock is longer than one second of a stationary clock. ***Hence a moving clock appears to run slow as compared to a stationary clock.***

iv) It should be noted that the phenomena of length-contraction and time-dilation are reciprocal.

A metre scale lying in the frame S' appears to have contracted to the observer in the frame S. Similarly a metre scale lying in the frame S appears to have contracted to the observer in the frame S' by the same amount.

A clock lying in the frame S' appears to be running slow to the observer in the frame S. Similarly a clock lying in the frame S appears to be running slow to the observer in the frame S' in the same way.

2.14 VELOCITY OF LIGHT 'C' AS A LIMITING VELOCITY IN THE UNIVERSE :

$$\Delta x = \sqrt{1 - \frac{V^2}{c^2}}\,(\Delta x)_0, \qquad \Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

If $V = c$, then $\sqrt{1 - \frac{V^2}{c^2}} = 0$. Hence $\Delta x = 0$ and $\Delta t = \infty$.

i.e. the length of the body would appear to be zero and time-interval would appear to be infinite. This is not possible in nature. Hence a body cannot move with a velocity V equal to the velocity of light c in free space.

Moreover if $V > c$ then $\frac{V^2}{c^2} > 1$,

$1 - \frac{V^2}{c^2}$ becomes negative and $\sqrt{1 - \frac{V^2}{c^2}}$ becomes imaginary. But then Δx and Δt also become imaginary. Even this is not possible in nature. Hence a body cannot move with a velocity V greater than the velocity of light c.

Thus a real material body can neither attain the velocity of light nor travel faster than light. All the velocities which it can attain in reality are always less than c.

Only the electromagnetic waves, including light, travel with velocity c in free space. (i.e. only photons can travel with velocity c in free space). As such the velocity of light c in free space becomes the highest possible velocity in nature.

Thus the velocity of light $c = 3 \times 10^8$ m/s in free space is the limiting velocity in the universe.

A direct consequence of this result is that in the universe, any effect, any signal, any message, any action can be transmitted from one place to another place only with a **finite velocity** which is less than or equal to c. Instantaneous transmission with infinite velocity is impossible in nature, according to the Special theory of Relativity.

We have seen in the First chapter that Faraday introduced the concept of field in Physics in order to explain "action at a distance" from the electric charges and magnetic poles. Now, according to the electromagnetic theory, this action at a distance occurs in the fields and forces are transmitted in the fields **instantaneously** i.e. with infinite velocity.

According to the Special theory of Relativity this is absurd. The action at a distance occurs and forces are transmitted in any field (gravitational, electric or magnetic field) with a finite velocity which may be less than or at the most equal to the velocity of light $c = 3 \times 10^8$ m/s. Even the disturbances produced in electric and magnetic fields propagate in the respective fields with a finite velocity which can be less than or at the most equal to c. Thus this theory of Relativity infuses life and reality into the Faraday's concept of field. Einstein says : "The electromagnetic field is, for a modern physicist, as real as the chair on which he sits".

2.15 RIGID BODY IN RELATIVITY :

In Classical Mechanics a rigid body is defined as that body in which the distance between any two particles does not change under the action of any force, however large.

When a force is applied at a point on a rigid body, the force must be transmitted to all other particles instantaneously so that all particles would start moving simultaneously because the distance between any two particles of the rigid body should remain the same.

In Relativity, however, this is not possible because the force is transmitted to the other particles with a **finite velocity** and not instantaneously. Hence all particles of the rigid body are set into motion at different instants of time and the body suffers a deformation. Thus in Relativity, no body is absolutely a rigid body.

2.16 EXPERIMENTAL VERIFICATION OF TIME-DILATION :

In order to observe the dilation of time experimentally the following are the requirements :

i) Particles travelling with very high velocity V comparable to the velocity of light c,

ii) Small time-intervals so that the distance travelled with high velocity would be small and easily measurable.

Some elementary particles like π - mesons (pions), μ - mesons (muons) etc. satisfy these requirements.

A muon decays into an electron, on an average, 2×10^{-6} sec. after it is produced in nuclear reactions in the laboratory. This is the time-interval between two events, viz. birth and death of muons. When muons are stationary this time-interval can be regarded as proper time-interval i.e. $(\Delta t)_0 = 2 \times 10^{-6}$ sec.

Now, muons are also produced in upper layers of the atmosphere by the cosmic rays arriving at the earth from outer space. These muons travel with very high velocity equal to $V = 0.99c$. If we take the life-time of these moving muons as $(\Delta t)_0 = 2 \times 10^{-6}$ sec. then the distance travelled by them is given by

$$\begin{aligned} s &= V \times (\Delta t)_0 \\ &= 0.99\, c \times 2 \times 10^{-6} \\ &= 594 \text{ m.} \end{aligned}$$

which is even less than a kilometre. But actually these muons are created at attitudes which are about 10 times greater than this value.

We are getting unconvincing result because the life-time of moving muons is not $(\Delta t)_0$ but Δt such that

$$\begin{aligned} \Delta t &= \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}} \\ &= \frac{2 \times 10^{-6}}{\sqrt{1 - (0.99)^2}} = 14.17 \times 10^{-6} \text{ sec.} \end{aligned}$$

The distance covered by muons in this time Δt is given by

$$\begin{aligned} s &= V . \Delta t. \\ &= 0.99\, c \times 14.17 \times 10^{-6} = 4208\text{m} = 4.208 \text{ km} \end{aligned}$$

Thus muons are created at an altitude of 4.208 km above the sea level. This result is quite acceptable. Thus time - dilation has to be taken into consideration while dealing with the particles travelling with very high velocity comparable to the velocity of light in high energy Physics and engineering.

Problem : A metre stick is projected into space at such a speed that its length appears contracted to only 50 cm. Find out the speed of the stick.

Given : $(\Delta x)_0 = 1\text{m}$, $\Delta x = 50 \text{ cm} = 0.5 \text{ m}$ $V = ?$

Solution :

$$\therefore \Delta x = (\Delta x)_0 \sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore 0.5 = 1\sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore \sqrt{1-\frac{V^2}{c^2}} = 0.5 = \frac{1}{2}$$

$$\therefore 1-\frac{V^2}{c^2} = \frac{1}{4}$$

$$\therefore \frac{V^2}{c^2} = \frac{3}{4}$$

$$\therefore V = \sqrt{\frac{3}{4}}\,c = 0.866\,c.$$

$\therefore$ Speed of the stick = 0.866 *c*.

Problem : A pion has average life of 10^{-7} sec which is measured when it is at rest. What average distance the pion would travel before decaying in a laboratory frame in which its velocity is 0.6 *c*? Compare this distance with the distance which the pion sees travelling through by itself. Take $c = 3 \times 10^8$ m/s

Solution :

$(\Delta t)_0 = 10^{-7}$ sec., $V = 0.6\,c$

In the laboratory frame, the life of moving pion is given by

$$\therefore \Delta t = \frac{(\Delta t)_0}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{10^{-7}}{\sqrt{1-0.36}} = \frac{10^{-7}}{0.8} = 1.25 \times 10^{-7} \text{ sec.}$$

The distance covered by the pion in this time is given by

$$s = V\Delta t.$$

$$= 0.6\,c \times 1.25 \times 10^{-7}$$

$$= 0.6 \times 3 \times 10^8 \times 1.25 \times 10^{-7} = 22.5 \text{ m.}$$

$\therefore$ Average distance covered by the pion relative to the laboratory frame = 22.5 m.

This distance is stationary in the laboratory frame. Hence $(\Delta x)_0 =$ 22.5 m. For the pion it is moving in the backward direction. Hence the distance covered by the pion as measured by the pion itself is given by

$$\Delta x = (\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}} = 22.5\sqrt{1 - 0.36} = 22.5 \times 0.8 = 18 \text{ m}$$

∴ Average distance covered by pion as seen by the pion itself = 18m.

Problem : A muon formed high up in the atmosphere travels with a speed 0.99c for a distance 5.4 km before it decays. What is the life of muon as measured by us and as measured by the muon? How much distance does the muon travel as measured by itself?

Solution :

$V = 0.99\, c. \qquad s = 5.4 \text{ km} = 5400\text{m}$

Now, $V = \frac{s}{t} \qquad \therefore t = \frac{s}{V}$

∴ Life of muon as measured by us $= \dfrac{5400}{0.99 \times 3 \times 10^8}$

$= 1.818 \times 10^{-5}$ sec.

Since the muon is moving relative to us its birth and decay occur at different places. i.e. the place of events is moving.

$\therefore \Delta t = 1.818 \times 10^{-5}$ sec.

For muon these events occur at the same place i.e. in muon itself. Hence life of muon as measured by muon itself is $(\Delta t)_0$.

$$\text{Now, } \Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore (\Delta t)_0 = \Delta t \sqrt{1 - \frac{V^2}{c^2}}$$

$$= 1.818 \times 10^{-5}\sqrt{1 - (0.99)^2}$$

$$= 1.818 \times \sqrt{0.0199} \times 10^{-5} = 0.2565 \times 10^{-5} \text{ sec}$$

∴ Life of muon as measured by the muon $= 2.565 \times 10^{-6}$ sec.

Distance travelled by muon as measured by us = 5400 m

This distance is at rest for us but for muon it is moving in the backward direction.

$\therefore (\Delta x)_0 = 5400$ m.

∴ Distance travelled by muon as measured by muon is given by

$$\Delta x = (\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}}$$

$$= 5400 \sqrt{1 - (0.99)^2} = 5400 \sqrt{0.0199} = 761.8 \text{ m}$$

Problem : A stick at rest in the frame S has a length L. It lies in the X-Y plane making an angle θ with the X-axis. Find its length L' and inclination to the X'-axis as measured by an observer in S' frame which moves with a constant velocity V along X-axis.

Solution :

Relative to the frame S,

Component of length L along X-axis = L cosθ

Component of length L along Y-axis = L sinθ

Since these lengths are proper lengths,

$$(\Delta x)_0 = L \cos\theta,$$

$$(\Delta y)_0 = L \sin\theta.$$

The corresponding observed lengths in the frame S' are given by

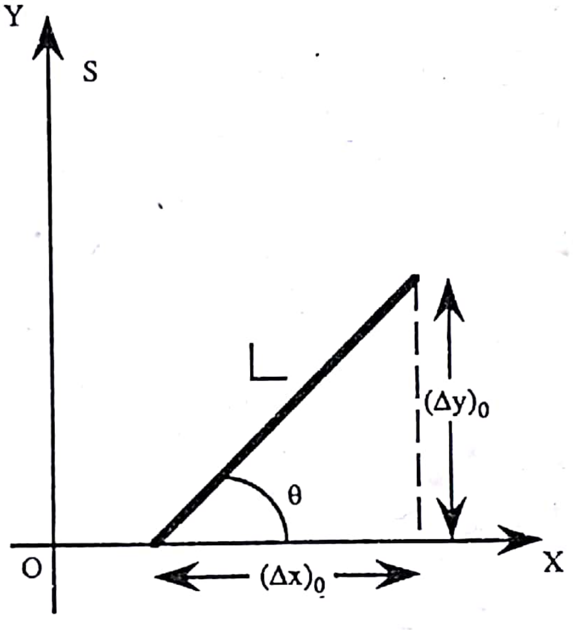


Fig. 2.9 Stick in X-Y plane

$$\Delta x = (\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}} = L \cos\theta \sqrt{1 - \frac{V^2}{c^2}}$$

$$\Delta y = (\Delta y)_0 = L \sin\theta.$$

Let the length of the stick relative to the frame S' be L'.

$$\therefore L'^2 = (\Delta x)^2 + (\Delta y)^2$$

$$= L^2 \cos^2\theta \left(1 - \frac{V^2}{c^2}\right) + L^2 \sin^2\theta$$

$$= L^2 \cos^2\theta - L^2 \cos^2\theta \frac{V^2}{c^2} + L^2 \sin^2\theta$$

$$= L^2 - L^2 \cos^2\theta \frac{V^2}{c^2}$$

$$= L^2 \left[1 - \frac{V^2}{c^2} \cos^2\theta\right]$$

$$\therefore L' = L\sqrt{1 - \frac{V^2}{c^2} \cos^2\theta}$$

This is the length of the stick as observed in the frame S'.

Relative to the frame S,

$$\tan\theta = \frac{(\Delta y)_0}{(\Delta x)_0}$$

Let the inclination of the stick to X' - axis be θ'.

$$\therefore \tan\theta' = \frac{\Delta y}{\Delta x}$$

$$= \frac{(\Delta y)_0}{(\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{(\Delta y)_0}{(\Delta x)_0} \cdot \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{\tan\theta}{\sqrt{1 - \frac{V^2}{c^2}}}$$

This formula gives the inclination θ' of the stick to X'-axis in the frame S'.

Problem : Prove that the volume element $dx.dy.dz.dt$ in four-dimensional space (i.e. Minkowski space-time) is invariant under Lorentz transformations.

Solution :

By Lorentz-Fitzgerald length -contraction,

$$dx = (dx)_0 \sqrt{1 - \frac{V^2}{c^2}}, \quad dy = (dy)_0, \quad dz = (dz)_0$$

By Einstein's time - dilation ,

$$dt = \frac{(dt)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore dx.dy.dz.dt = (dx)_0 \sqrt{1 - \frac{V^2}{c^2}} \cdot (dy)_0 \cdot (dz)_0 \frac{(dt)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= (dx)_0 \cdot (dy)_0 \cdot (dz)_0 \cdot (dt)_0$$

This shows that the four-dimensional volume element is invariant under Lorentz transformations.

2.17 LORENTZ TRANSFORMATIONS OF VELOCITY :

Let us consider a particle moving in space. Let its velocity relative to the frame S be u having components u_x, u_y, u_z along X, Y and Z -axes respectively.

Then $u_x = \frac{dx}{dt} \qquad u_y = \frac{dy}{dt} \qquad u_z = \frac{dz}{dt}$

Let its corresponding velocity relative to the frame S' be u' having components u'_x, u'_y, u'_z along X', Y', Z' - axes respectively.

Then $u'_x = \frac{dx'}{dt'}, \qquad u'_y = \frac{dy'}{dt'}, \qquad u'_z = \frac{dz'}{dt'}$

i) $$u'_x = \frac{dx'}{dt'}$$

$$= \frac{d\left(\frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}\right)}{d\left(\frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right)}$$

$$= \frac{dx - V\,dt}{dt - \frac{V}{c^2}dx} = \frac{\frac{dx}{dt} - V}{1 - \frac{V}{c^2}\frac{dx}{dt}} = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}$$

ii) $$u'_y = \frac{dy'}{dt'}$$

$$= \frac{dy}{d\left(\frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right)}$$

$$= \frac{dy\sqrt{1 - \frac{V^2}{c^2}}}{dt - \frac{V}{c^2}dx} = \frac{\frac{dy}{dt}\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V}{c^2}\frac{dx}{dt}} = \frac{u_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$$

iii) Using the same method as in (ii) we can prove,

$$u'_z = \frac{u_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$$

Thus we have,

$$\boxed{u'_x = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}, \quad u'_y = \frac{u_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}, \quad u'_z = \frac{u_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}}$$

These are Lorentz transformation equations for velocity components.

Remarks :

i) As happens in Classical Mechanics, if the velocity u of the particle and the velocity V of the frame of reference S' are small in comparison with the velocity of light c in empty space, then $\frac{V^2}{c^2}$ and $\frac{Vu_x}{c^2}$ become negligible in comparison with 1. Hence the above equations reduce to

$$u'_x = u_x - V, \qquad u'_y = u_y, \qquad u'_z = u_z$$

These are nothing but the Galilean transformation equations for velocity components. (We have obtained them in § 1.10 of the first chapter).

ii) A ray of light is travelling with velocity c along X'-axis relative to the frame S'. (Fig. 2.10) .We shall find out its velocity relative to the frame S.

i.e. $u'_x = c$ and $u_x = ?$

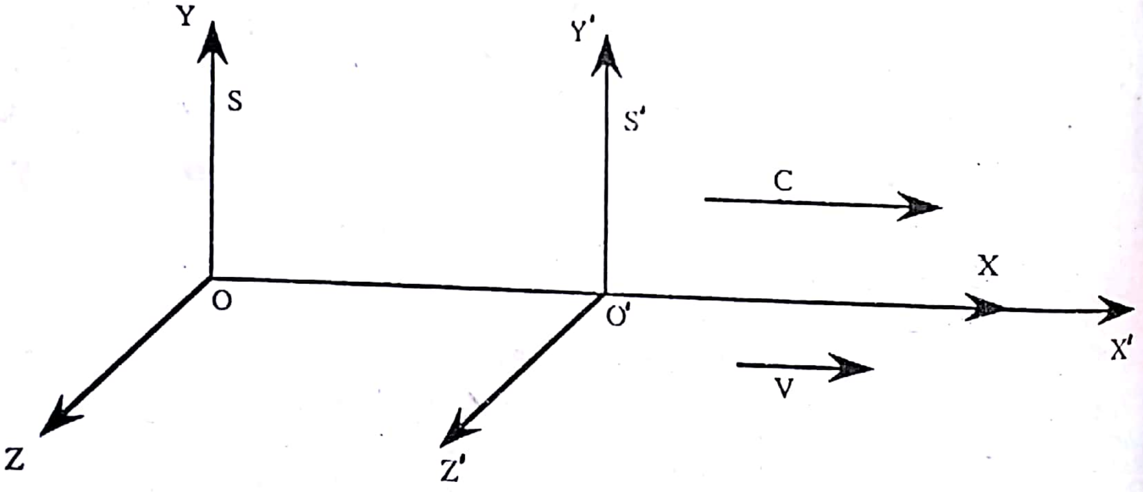


Fig. 2.10 A ray of light travelling with velocity c relative to the frame S'

By Galilean transformations of velocity,

$$u'_x = u_x - V$$

$$\therefore u_x = u'_x + V = c + V$$

which is greater than c! This is impossible according to the Special theory of Relativity.

By Lorentz transformations of velocity,

$$u'_x = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}$$

Taking inverse transformations,

$$u_x = \frac{u'_x + V}{1 + \frac{Vu'_x}{c^2}} = \frac{c + V}{1 + \frac{Vc}{c^2}} = \frac{c + V}{\frac{c + V}{c}} = c$$

i.e. in the frame S also the velocity of light is c. Thus the velocity of light remains the same in all inertial frames of reference. It does not depend upon the state of motion of the source or observer. Moreover, a velocity greater than c can not be achieved by introducing any frame of reference. Velocity greater than c does not exist in nature. The theory of Relativity is excellently consistent everywhere.

2.18 EINSTEIN'S LAW OF VELOCITY ADDITION :

We shall now obtain a formula representing Einstein's law of velocity addition. This formula, first devised by Einstein, is particularly very useful in solving problems on velocity transformations.

Let us consider a particle moving in space. Let its velocity relative to the frame S be u having component u_x along X - axis.

$$\therefore u_x = \frac{dx}{dt}.$$

Let its corresponding velocity relative to the frame S' be u' having component u'_x along X' - axis.

$$\therefore u'_x = \frac{dx'}{dt'}$$

$$\text{Now, } u'_x = \frac{dx'}{dt'}$$

$$= \frac{d\left(\frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}\right)}{d\left(\frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right)}$$

$$= \frac{dx - Vdt}{dt - \frac{V}{c^2}dx} = \frac{\frac{dx}{dt} - V}{1 - \frac{V}{c^2}\frac{dx}{dt}} = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}$$

$$\text{i.e. } u'_x = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}} \qquad \text{...(2.22)}$$

Let us introduce three inertial frames of reference (Fig. 2.11). The frame S' is moving with uniform linear velocity V relative to the frame S along X-axis. The frame S" is moving with uniform linear velocity W relative to S' along X'-axis. We shall find out the velocity of S" relative to S along X-axis.

i.e. in this case, $u'_x = W$ and $u_x = ?$

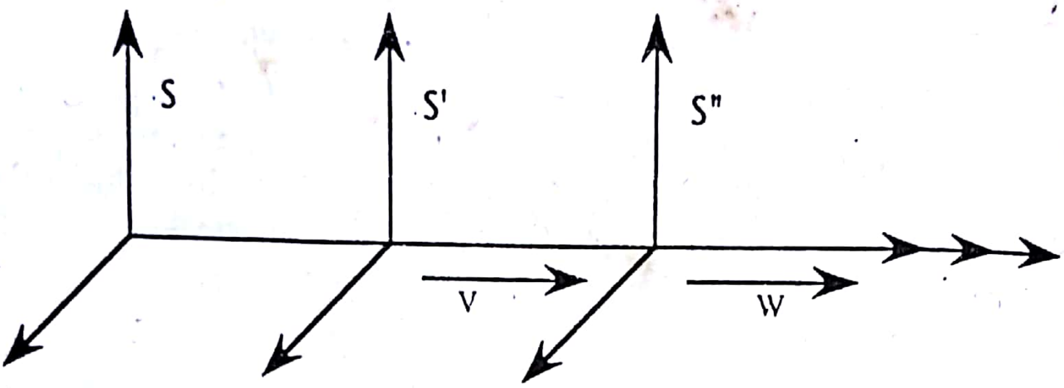


Fig. 2.11 Three inertial frames of reference

Taking inverse transformations of equation (2.22) we get

$$u_x = \frac{u'_x + V}{1 + \frac{Vu'_x}{c^2}} = \frac{W + V}{1 + \frac{VW}{c^2}}$$

Let us take $u_x = U$. Then the velocity U of the frame S" relative to S is given by

$$\boxed{U = \frac{V + W}{1 + \frac{VW}{c^2}}}$$

This equation represents Einstein's law of velocity addition .

Note : It should be noted that the velocity of the frame S" relative to the frame S is not simply V + W (as is the case in Classical Mechanics) but it is less than V + W.

Problem : Two electrons leave a radioactive sample in opposite directions, each having a speed 0.67 c relative to the sample. Find the velocity of one electron relative to the other electron.

Solution :

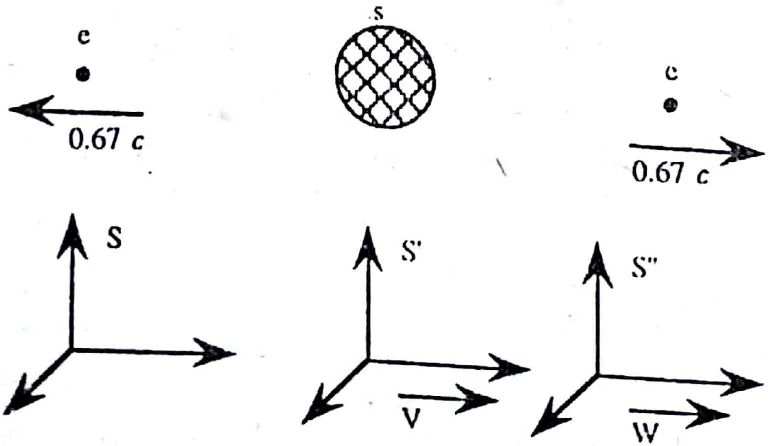


Fig. 2.12 Application of Einstein's law of velocity addition

Let us attach the frame S to one electron, S' to the sample and S" to the second electron as shown in Fig. 2.12.

Thus sample moves with velocity 0.67 c relative to the first electron i.e. relative to the frame S.

$$\therefore V = 0.67\ c.$$

The second electron moves with velocity 0.67 c relative to the sample i.e. relative to the frame S'.

$$\therefore W = 0.67\ c$$

We have to find velocity of the second electron relative to the first electron. i.e. U = ?

By Einstein's law of velocity addition,

$$U = \frac{V + W}{1 + \frac{VW}{c^2}}$$

$$= \frac{0.67\,c + 0.67\,c}{1 + \frac{0.67\,c \times 0.67\,c}{c^2}} = \frac{1.34\,c}{1 + (0.67)^2} = 0.9268\ c.$$

$\therefore$ Velocity of one electron relative to the other electron = 0.9268 c

Note : According to Galilean transformations of velocity, the velocity of one electron relative to the second electron would be = 0.67 c + 0.67 c = 1.34 c which is greater than c. This is absurd according to the theory of Relativity.

Problem : A man on the moon observes two spaceships coming towards him from opposite directions at speeds 0.8 c and 0.9 c respectively. What is the speed of one ship relative to the other ship?

Solution :

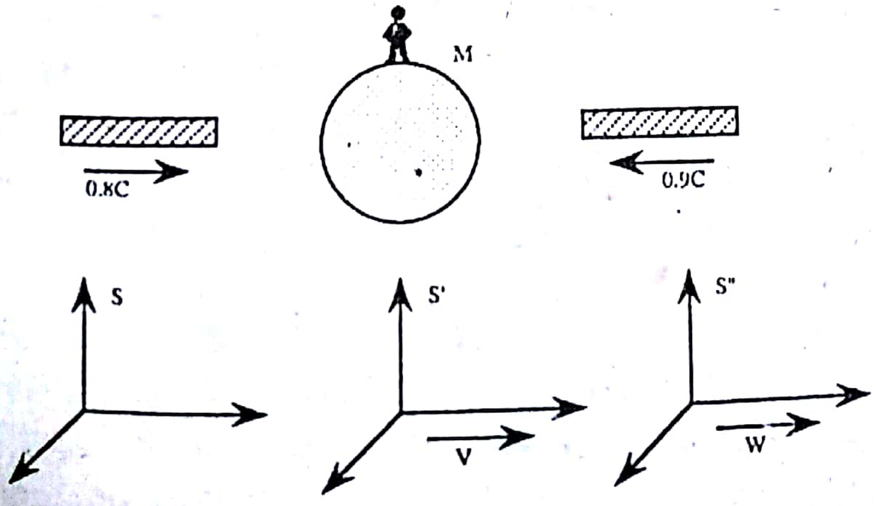


Fig. 2.13 Application of Einstein's law of velocity addition

Let us attach the frame S to one spaceship, S' to the moon and S" to the second spaceship as shown in Fig. 2.13.

Then $V = -0.8\,c$, $W = -0.9\,c$ and $U = ?$

By Einstein's law of velocity addition,

$$U = \frac{V+W}{1+\frac{VW}{c^2}}$$

$$= \frac{-0.8\,c - 0.9\,c}{1+\frac{0.8\,c \times 0.9\,c}{c^2}} = \frac{-1.7\,c}{1+0.72} = -\frac{1.7\,c}{1.72} = -0.9884\,c.$$

∴ Velocity of one spaceship relative to the second spaceship

$= -0.9884$ c.

The negative sign indicates that each spaceship is moving towards the other.

Problem : Two electrons approach each other with the same velocity, 0.7 *c* relative to an observer. Calculate their relative velocity.

Solution :

Let us attach the frame S to the observer, S' to one electron and S" to the second electron.

Then $V = 0.7\,c$, $U = -0.7\,c$ and $W = ?$

By Einstein's law of velocity addition,

$$U = \frac{V+W}{1+\frac{VW}{c^2}}$$

$$\therefore U + \frac{UVW}{c^2} = V + W$$

$$\therefore U - V = W\left[1 - \frac{UV}{c^2}\right]$$

$$\therefore W = \frac{U-V}{\left[1-\frac{UV}{c^2}\right]}$$

$$= \frac{-0.7\,c - 0.7\,c}{1+\frac{0.7\,c \times 0.7\,c}{c^2}}$$

$$= \frac{-1.4\,c}{1+0.49} = -\frac{1.4\,c}{1.49} = -0.9396\,c$$

∴ Velocity of one electron relative to the other electron

$= -0.9396\,c.$

Problem : The frame S' is moving with velocity V_1 relative to the frame S along X-axis. The frame S" is moving with velocity V_2 relative to the frame S' along X'-axis. If the velocity of S" relative to S is V then prove that

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 . \beta_2}$$

where β stands for $\frac{V}{c}$

Solution :

By Einstein's law of velocity addition,

$$U = \frac{V + W}{1 + \frac{VW}{c^2}}$$

In this case, $V = V_1$, $W = V_2$ and $U = V$.

$$\therefore V = \frac{V_1 + V_2}{1 + \frac{V_1 V_2}{c^2}} = \frac{V_1 + V_2}{1 + \frac{V_1}{c} . \frac{V_2}{c}}$$

Dividing both sides by c,

$$\frac{V}{c} = \frac{\frac{V_1}{c} + \frac{V_2}{c}}{1 + \frac{V_1}{c} . \frac{V_2}{c}}$$

$$\therefore \beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 . \beta_2}$$

This is the required expression.

2.19 FRESNEL'S DRAG FORMULA USING EINSTEIN'S LAW OF VELOCITY ADDITION :

In § 1.27 of the First chapter we have described Fizeau's experiment to detect the effect of the motion of the transmitting medium on the velocity of light. We have come to the conclusion that the experiment confirms the correctness of Fresnel's drag formula (1.18), namely,

$$v = \frac{c}{n} \pm v_w \left(1 - \frac{1}{n^2}\right).$$

The experiment rejected the formula given by the Galilean transformations of velocity. We shall now obtain Fresnel's drag formula using Einstein's law of velocity addition as follows :

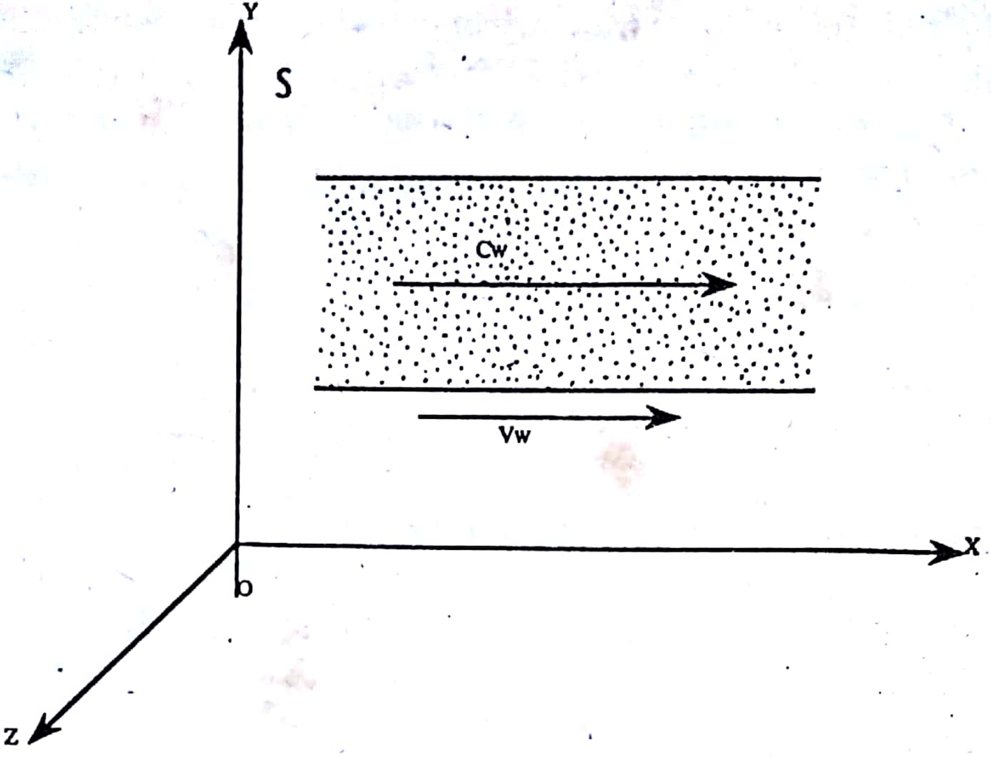


Fig. 2.14 A ray of light travelling in flowing liquid

A ray of light is travelling in a liquid (say, water) having refractive index n. Its velocity c_w relative to the liquid is given by

$$n = \frac{c}{c_w} \qquad \therefore c_w = \frac{c}{n}$$

The liquid is moving with velocity v_w, relative to the observer in the frame S. We shall find out the velocity v of the ray relative to the same observer in the frame S.

In order to apply Einstein's law of velocity addition, let us attach the frame S' to the liquid and the frame S" to the ray of light travelling in the liquid.(See Fig. 2.14)

$$\text{Then } V = v_w, \qquad W = c_w = \frac{c}{n} \text{ and } U = v.$$

∴ By Einstein's law of velocity addition,

$$U = \frac{V + W}{1 + \frac{VW}{c^2}}$$

$$\therefore v = \frac{v_w + \frac{c}{n}}{1 + \frac{v_w \frac{c}{n}}{c^2}}$$

$$= \frac{v_w + \frac{c}{n}}{1 + \frac{v_w}{cn}} = \left(v_w + \frac{c}{n}\right)\left(1 + \frac{v_w}{cn}\right)^{-1} \qquad ...(2.23)$$

By Binomial Theorem,

$$(a + x)^n = a^n + na^{n-1}.x + \frac{n(n-1)}{2} a^{n-2}.x^2 + ...$$

$$\therefore \left(1 + \frac{v_w}{cn}\right)^{-1} = 1^{-1} + (-1)(1)^{-2}\left(\frac{v_w}{cn}\right) + \frac{(-1)(-2)}{2}(1)^{-3}\left(\frac{v_w}{cn}\right)^2 + ...$$

$$= 1 - \frac{v_w}{cn} + \left(\frac{v_w}{cn}\right)^2 + \text{terms in higher powers of } \left(\frac{v_w}{cn}\right)$$

Now, v_w is small in comparison with c. Hence $\frac{v_w}{cn}$ is much less than 1. Due to this, the terms in higher power of $\frac{v_w}{cn}$ become negligible in comparison with the first two terms on R.H.S.

$$\left(1 + \frac{v_w}{cn}\right)^{-1} = 1 - \frac{v_w}{cn}.$$

Hence equation (2.23) reduces to

$$v = \left(v_w + \frac{c}{n}\right)\left(1 - \frac{v_w}{cn}\right) = v_w - \frac{v_w^2}{cn} + \frac{c}{n} - \frac{v_w}{n^2}$$

Even $\frac{v_w^2}{cn}$ becomes negligible in comparison with the other terms.

$$\therefore v = v_w + \frac{c}{n} - \frac{v_w}{n^2} = \frac{c}{n} + v_w\left(1 - \frac{1}{n^2}\right)$$

In this case, the liquid travels in the same direction as the ray. If we reverse the direction of motion of the liquid then v_w becomes negative and the above equation becomes,

$$v = \frac{c}{n} - v_w\left(1 - \frac{1}{n^2}\right)$$

Thus we have,

$$v = \frac{c}{n} \pm v_w\left(1 - \frac{1}{n^2}\right)$$

This is nothing but Fresnel's drag formula. Thus Einstein's law of velocity addition leads to Fresnel's drag formula in a very natural and decisive way.

2.20 LORENTZ TRANSFORMATIONS OF ACCELERATION :

Let us consider a particle moving in space. Let its acceleration relative to the frame S be a having components a_x, a_y, a_z along X, Y and Z -axes respectively.

$$\text{Then } a_x = \frac{du_x}{dt}, \quad a_y = \frac{du_y}{dt}, \quad a_z = \frac{du_z}{dt}.$$

Let its corresponding acceleration relative to the frame S' be a' having components a'_x, a'_y, a'_z, along X',Y', and Z' - axes respectively.

$$\text{Then } a'_x = \frac{du'_x}{dt'}, \quad a'_y = \frac{du'_y}{dt'}, \quad a'_z = \frac{du'_z}{dt'}$$

i) $$du'_x = d\frac{u_x - V}{1 - \frac{V.u_x}{c^2}}$$

$$= \frac{\left(1 - \frac{V.u_x}{c^2}\right) d(u_x - V) - (u_x - V)\, d\left(1 - \frac{V.u_x}{c^2}\right)}{\left(1 - \frac{V.u_x}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{V.u_x}{c^2}\right)(du_x) - (u_x - V)\left(-\frac{V}{c^2} du_x\right)}{\left(1 - \frac{V.u_x}{c^2}\right)^2}$$

$$= \frac{\left(1 - \frac{V.u_x}{c^2}\right) du_x + (u_x - V)\frac{V}{c^2} du_x}{\left(1 - \frac{V.u_x}{c^2}\right)^2}$$

$$= \frac{du_x\left(1 - \frac{V.u_x}{c^2} + \frac{V.u_x}{c^2} - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V.u_x}{c^2}\right)^2} = \frac{du_x\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{V.u_x}{c^2}\right)^2}$$

$$dt' = d\left(\frac{t - \frac{V.x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right)$$

$$= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(dt - \frac{V}{c^2} dx\right)$$

$$= \frac{dt}{\sqrt{1 - \frac{V^2}{c^2}}}\left(1 - \frac{V}{c^2}\frac{dx}{dt}\right) = \frac{dt}{\sqrt{1 - \frac{V^2}{c^2}}}\left(1 - \frac{Vu_x}{c^2}\right)$$

$$\therefore a'_x = \frac{du'_x}{dt'}$$

$$= \frac{du_x\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V.u_x}{c^2}\right)^2} \times \frac{\sqrt{1-\frac{V^2}{c^2}}}{dt\left(1-\frac{Vu_x}{c^2}\right)}$$

$$= \frac{du_x}{dt}\frac{\left(1-\frac{V^2}{c^2}\right)^{3/2}}{\left(1-\frac{V.u_x}{c^2}\right)^3}$$

$$= a_x\frac{\left(1-\frac{V^2}{c^2}\right)^{3/2}}{\left(1-\frac{V.u_x}{c^2}\right)^3}$$

ii) $$du'_y = d\frac{u_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{Vu_x}{c^2}}$$

$$= \sqrt{1-\frac{V^2}{c^2}}\left[d\frac{u_y}{1-\frac{V.u_x}{c^2}}\right]$$

$$= \sqrt{1-\frac{V^2}{c^2}}\left[\frac{\left(1-\frac{Vu_x}{c^2}\right)du_y - u_y\, d\left(1-\frac{Vu_x}{c^2}\right)}{\left(1-\frac{Vu_x}{c^2}\right)^2}\right]$$

$$= \frac{\sqrt{1-\frac{V^2}{c^2}}}{\left(1-\frac{Vu_x}{c^2}\right)^2}\left[\left(1-\frac{Vu_x}{c^2}\right)du_y - u_y\left(-\frac{V}{c^2}du_x\right)\right]$$

$$= \frac{\sqrt{1-\frac{V^2}{c^2}}}{\left(1-\frac{V.u_x}{c^2}\right)^2}\left[\left(1-\frac{Vu_x}{c^2}\right)du_y + \frac{Vu_y}{c^2}du_x\right]$$

$$\therefore a'_y = \frac{du'_y}{dt'}$$

$$= \frac{\sqrt{1-\frac{V^2}{c^2}}}{\left(1-\frac{V.u_x}{c^2}\right)^2}\left[\left(1-\frac{V.u_x}{c^2}\right)du_y + \frac{V.u_y}{c^2}du_x\right] \times \frac{\sqrt{1-\frac{V^2}{c^2}}}{dt\left(1-\frac{V.u_x}{c^2}\right)}$$

$$= \frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{Vu_x}{c^2}\right)^3}\left[\left(1-\frac{V.u_x}{c^2}\right)\frac{du_y}{dt}+\frac{Vu_y}{c^2}\frac{du_x}{dt}\right]$$

$$= \frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{Vu_x}{c^2}\right)^3}\left[\left(1-\frac{V.u_x}{c^2}\right)a_y+\frac{Vu_y}{c^2}a_x\right]$$

iii) Using the same method as in (ii) we can prove

$$a'_z = \frac{\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{Vu_x}{c^2}\right)^3}\left[\left(1-\frac{V.u_x}{c^2}\right)a_z+\frac{Vu_z}{c^2}a_x\right]$$

In (i), (ii) and (iii) above we have obtained Lorentz transformations of the components of acceleration.

Remarks : In Newtonian Relativity, $a'_x = a_x, a'_y = a_y, a'_z = a_z$ i.e. the acceleration is invariant under Galilean transformations (See § 1.11 of the First chapter). This is not so in Special theory of Relativity. The acceleration is a relative quantity and varies from one inertial frame to another inertial frame.

However, you can see easily, for small values of u and V the Lorentz transformation equations of acceleration reduce to Galilean transformation equations of acceleration.

2.21 RELATIVISTIC THEORY OF ABERRATION OF LIGHT AND DOPPLER EFFECT :

A simple harmonic progressive wave travelling along X-axis is given by

$$z = A \sin 2\pi\left[\frac{t}{T}-\frac{x}{\lambda}\right] = A \sin 2\pi\left[\upsilon t-\frac{x}{\lambda}\right]$$

Similarly a simple harmonic progressive wave travelling along Y-axis is given by

$$z = A \sin 2\pi\left[\upsilon t-\frac{y}{\lambda}\right]$$

From these equations we can compute the equation of a plane wave travelling in X-Y plane with its ray (or wavenormal) making an angle θ with X-axis (Fig. 2.15) as follows :

$$z = A\sin 2\pi\left[\upsilon t - \frac{x\cos\theta + y\sin\theta}{\lambda}\right]$$

$$z = A\sin 2\pi\left[\upsilon t - \frac{x\cos\theta}{\lambda} - \frac{y\sin\theta}{\lambda}\right] \qquad ...(2.24)$$

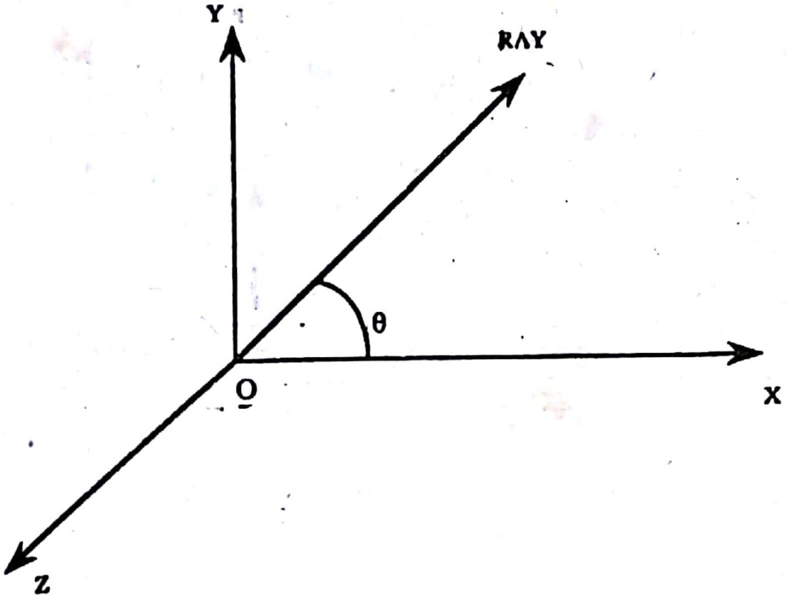


Fig. 2.15 Ray or wavenormal making an angle θ with X - axis

This equation represents the propagation of a plane wave relative to the frame S. According to the first postulate of the Special theory of Relativity, the equation representing the propagation of the same plane wave relative to the frame S' should have the same form. Hence it can be written as

$$z' = A'\sin 2\pi\left[\upsilon' t' - \frac{x'\cos\theta'}{\lambda'} - \frac{y'\sin\theta'}{\lambda'}\right] \qquad ...(2.25)$$

By Lorentz transformations of space and time co-ordinates,

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{V.x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Using these transformation equations, we shall transform the equation (2.25) from the frame S' to S as follows :

$$z' = A'\sin 2\pi\left[\upsilon' \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} - \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}\,\frac{\cos\theta'}{\lambda'} - \frac{y\sin\theta'}{\lambda'}\right]$$

$$= A' \sin 2\pi \left[\frac{\upsilon' t}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{\upsilon' \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} - \frac{x \cos\theta'}{\lambda' \sqrt{1-\frac{V^2}{c^2}}} \right.$$

$$\left. + \frac{Vt\cos\theta'}{\lambda'\sqrt{1-\frac{V^2}{c^2}}} - \frac{y\sin\theta'}{\lambda'} \right]$$

$$= A' \sin 2\pi \left[\frac{t}{\sqrt{1-\frac{V^2}{c^2}}} \left(\upsilon' + \frac{V\cos\theta'}{\lambda'} \right) \right.$$

$$\left. - \frac{x}{\sqrt{1-\frac{V^2}{c^2}}} \left(\frac{\upsilon' V}{c^2} + \frac{\cos\theta'}{\lambda'} \right) - \frac{y\sin\theta'}{\lambda'} \right]$$

This equation should be the same as equation (2.24). Hence equating the co-efficients of t, x and y in these equations, we get respectively,

$$\upsilon = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \left(\upsilon' + \frac{V\cos\theta'}{\lambda'} \right) \qquad \text{...(2.26)}$$

$$\frac{\cos\theta}{\lambda} = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \left(\frac{\upsilon' V}{c^2} + \frac{\cos\theta'}{\lambda'} \right) \qquad \text{...(2.27)}$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'} \qquad \text{...(2.28)}$$

Relativistic equation for Aberration of Light :

Dividing equation (2.28) by equation (2.27),

$$\frac{\frac{\sin\theta}{\lambda}}{\frac{\cos\theta}{\lambda}} = \frac{\sin\theta'}{\lambda'} \frac{\sqrt{1-\frac{V^2}{c^2}}}{\left(\frac{\upsilon' V}{c^2} + \frac{\cos\theta'}{\lambda'} \right)}$$

$$\therefore \quad \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta'}{\lambda'} \frac{c^2 \lambda' \sqrt{1-\frac{V^2}{c^2}}}{\left(\upsilon' \lambda' V + c^2 \cos\theta' \right)}$$

Now $\upsilon'\lambda' = c' = c$ by the second postulate of the Special theory of Relativity.

$$\therefore \tan\theta = \frac{\sin\theta' . c^2 \sqrt{1 - \frac{V^2}{c^2}}}{\left(cV + c^2\cos\theta'\right)}$$

$$= \frac{\sin\theta' \sqrt{1 - \frac{V^2}{c^2}}}{\left(\frac{V}{c} + \cos\theta'\right)}$$

Taking inverse transformations,

$$\tan\theta' = \frac{\sin\theta \sqrt{1 - \frac{V^2}{c^2}}}{\left(\cos\theta - \frac{V}{c}\right)} \qquad \text{...(2.29)}$$

This is relativistic equation for the aberration of light.

Relativistic equation for Doppler Effect :

Considering the equation (2.26)

$$\upsilon = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}\left(\upsilon' + \frac{V\cos\theta'}{\lambda'}\right)$$

$$\upsilon = \frac{\upsilon'}{\sqrt{1 - \frac{V^2}{c^2}}}\left(1 + \frac{V\cos\theta'}{\upsilon'\lambda'}\right)$$

Again, $\upsilon'\lambda' = c$

$$\therefore \upsilon = \frac{\upsilon'}{\sqrt{1 - \frac{V^2}{c^2}}}\left(1 + \frac{V}{c}\cos\theta'\right)$$

Taking inverse transformations,

$$\upsilon' = \frac{\upsilon}{\sqrt{1 - \frac{V^2}{c^2}}}\left(1 - \frac{V}{c}\cos\theta\right)$$

Now, let the source of light be fixed at the origin O' of the frame S' and the observer at the origin O of the frame S.

Since the source of light is stationary in the frame S', υ' is the proper frequency of light emitted by the source. We shall denote it by υ_0. For the observer in the frame S the source is moving with velocity V along X-axis. Hence υ is the **observed frequency** or non-proper frequency of light emitted by the source. The above equation becomes,

$$\nu_0 = \frac{\nu}{\sqrt{1-\frac{V^2}{c^2}}}\left(1-\frac{V}{c}\cos\theta\right)$$

$$\therefore \nu = \frac{\nu_0\sqrt{1-\frac{V^2}{c^2}}}{\left(1-\frac{V}{c}\cos\theta\right)} \qquad \text{...(2.30)}$$

This is the relativistic equation for Doppler effect. It shows that the observed frequency of light ν depends not only on the relative velocity V of the source and the observer but also on the angle θ at which the ray of light is received by the observer in the frame S. i.e. on the line of sight of the observer.

Two important cases develop when the line of sight is parallel and when it is perpendicular to the direction of relative motion between the source and the observer. They are as follows :

i) Longitudinal Doppler Effect :

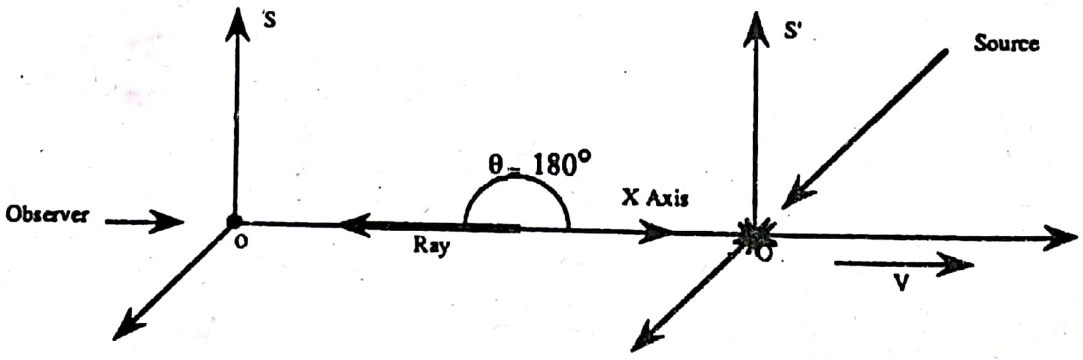


Fig. 2.16 Longitudinal Doppler effect

When the line of sight is parallel to the direction of the relative motion between the source and the observer, the observer receives the ray from the source along X - axis as shown in Fig. 2.16. The ray makes an angle $\theta = 180°$ with X - axis. Hence the equation (2.30) becomes,

$$\nu = \frac{\nu_0\sqrt{1-\frac{V^2}{c^2}}}{\left(1-\frac{V}{c}\cos 180°\right)}$$

$$= \frac{\upsilon_0 \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c}}$$

$$= \frac{\upsilon_0 \sqrt{c^2 - V^2}}{c + V}$$

$$= \frac{\upsilon_0 \sqrt{(c - V)(c + V)}}{(c + V)} = \upsilon_0 \sqrt{\frac{c - V}{c + V}}$$

$$\boxed{\upsilon = \upsilon_0 \sqrt{\frac{c - V}{c + V}},} \qquad \upsilon < \upsilon_0$$

In this case, the source is moving away from the observer or equivalently the observer is moving away from the source and the observed frequency of light is less than the proper frequency.

If the source moves towards the observer or observer moves towards the source then V becomes negative. Hence the above formula becomes,

$$\boxed{\upsilon = \upsilon_0 \sqrt{\frac{c + V}{c - V}},} \qquad \upsilon > \upsilon_0$$

i.e. when the source and observer move **towards** each other the observed frequency of light is **greater** than the proper frequency.

Thus when the line of sight is parallel to the direction of the relative motion between the source and the observer, the observed frequency of light comes out to be different from the original proper frequency of the light emitted by the source. This phenomenon is called Longitudinal Doppler effect.

ii) Transverse Doppler Effect :

When the line of sight is perpendicular to the direction of the relative motion between the source and the observer, the observer receives the ray from the source along Y- axis as shown in Fig. 2.17. The ray makes an angle $\theta = 90°$ with X - axis. Hence equation (2.30) becomes,

$$\upsilon = \frac{\upsilon_0 \sqrt{1 - \frac{V^2}{c^2}}}{\left(1 - \frac{V}{c} \cos 90°\right)} = \upsilon_0 \sqrt{1 - \frac{V^2}{c^2}}$$

$$\boxed{\upsilon = \upsilon_0 \sqrt{1 - \frac{V^2}{c^2}},} \qquad \upsilon < \upsilon_0$$

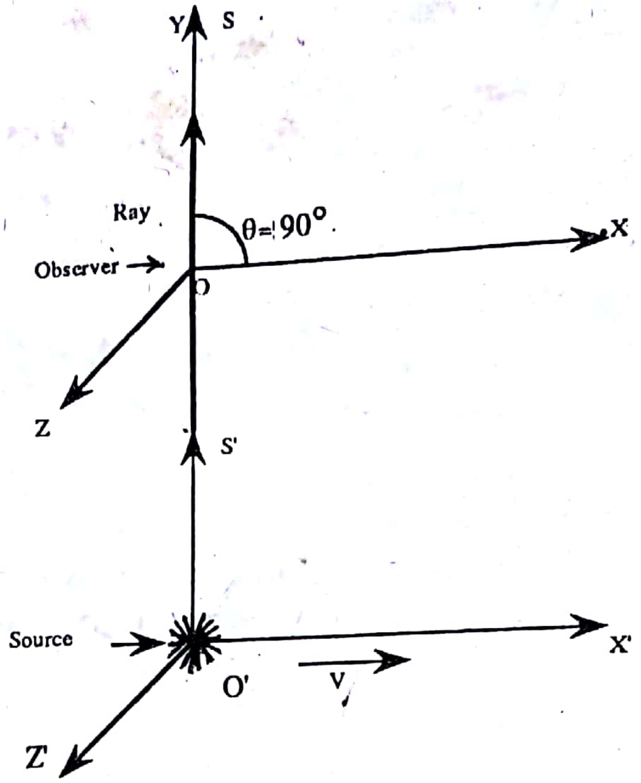


Fig. 2.17 Transverse Doppler Effect

Thus when the line of sight is perpendicular to the direction of relative motion between the source and the observer, the observed frequency of light comes out to be less than the original proper frequency of light emitted by the source. This phenomenon is called Transverse Doppler effect. Such transverse Doppler effect does not exist in Classical Physics at all.

Remark : Transverse Doppler effect can be very easily understood from the phenomenon of time -dilation. Frequency is given by

$$\upsilon = \frac{1}{T}$$

Now, moving time-intervals appear to be dilated i.e. expanded. Hence the period T appears to have increased and the frequency υ appears to have decreased due to the motion of the source.

2.22 CLASSICAL FORMULA FOR ABERRATION OF LIGHT FROM RELATIVISTIC FORMULA :

In the last section we have obtained the relativistic equation (2.29) for the aberration of light, namely,

$$\tan\theta' = \frac{\sin\theta\sqrt{1-\frac{V^2}{c^2}}}{\cos\theta - \frac{V}{c}}$$

Using approximations, we shall obtain the corresponding classical formula in what follows :

Let us attach the frame S to the overhead star and the frame S' to the earth which is moving with velocity V relative to the star due to its orbital motion. (Fig. 2.18).

The rays of light from the star travel in vertically downward direction relative to the frame S as shown in Fig. 2.18.

$$\therefore \theta = \frac{3\pi}{2}$$

Hence the relativistic formula becomes

$$\tan\theta' = \frac{\sin\frac{3\pi}{2}\sqrt{1-\frac{V^2}{c^2}}}{\cos\frac{3\pi}{2}-\frac{V}{c}}$$

$$= \frac{-1\sqrt{1-\frac{V^2}{c^2}}}{0-\frac{V}{c}}$$

$$= \frac{c}{V}\sqrt{1-\frac{V^2}{c^2}}$$

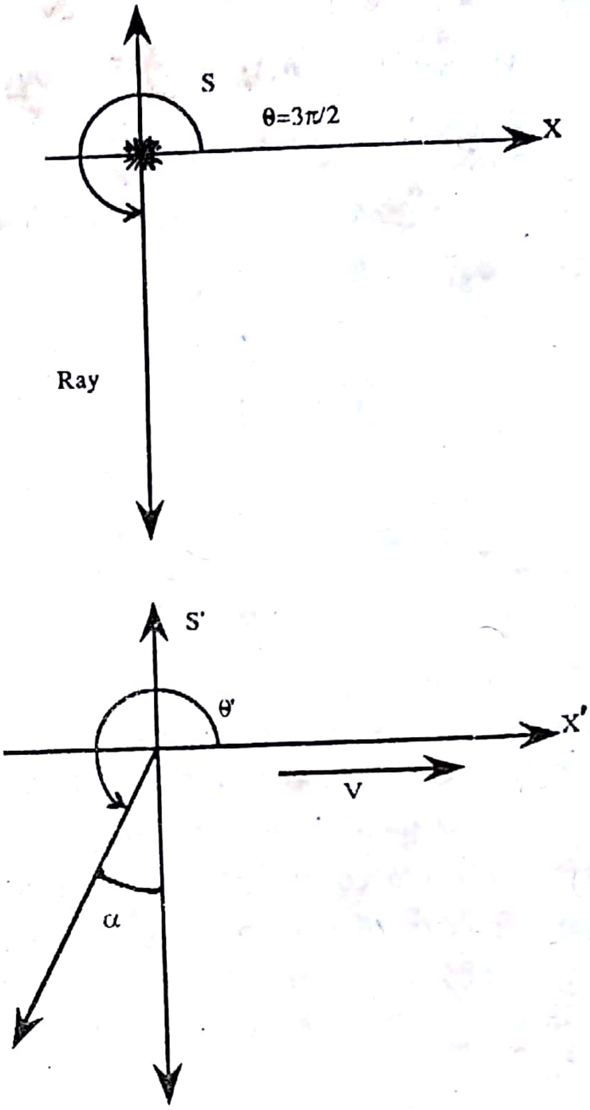


Fig 2.18 Relativistic explanation of aberration light

For approximations, we can consider V to be much smaller than c. i.e. $V << c$. Hence $\frac{V}{c} << 1$ and $\frac{V^2}{c^2}$ becomes negligible in comparison with 1. The above formula reduces to

$$\tan\theta' = \frac{c}{V} \qquad ...(2.31)$$

This equation shows that tan θ' is very large but finite. Now we know, $\tan\frac{3\pi}{2} = \infty$. Hence θ' must be slightly less than $\frac{3\pi}{2}$ as shown in Fig. 2.18. Therefore there must exist an angle α such that

$$\theta' + \alpha = \frac{3\pi}{2}$$

i.e. $\theta' = \frac{3\pi}{2} - \alpha$

Hence the equation (2.31) becomes,

$$\tan\left(\frac{3\pi}{2} - \alpha\right) = \frac{c}{V}$$

$$\therefore \frac{\sin\left(\frac{3\pi}{2} - \alpha\right)}{\cos\left(\frac{3\pi}{2} - \alpha\right)} = \frac{c}{V}$$

$$\therefore \frac{-\cos\alpha}{-\sin\alpha} = \frac{c}{V} \qquad \therefore \frac{\sin\alpha}{\cos\alpha} = \frac{V}{c}$$

$$\therefore \tan\alpha = \frac{V}{c}$$

This is exactly same as the equation (1.16) of §1.23 of the First chapter. Thus the Relativistic formula for aberration of light leads to the classical formula under the approximations of small velocities.

According to the Relativistic theory, due to the motion of the earth relative to the star, the rays of light from the star arrive at the earth in the direction making an angle α with the vertical direction (Fig 2.18). In order to receive them the telescope has to be kept inclined in the same direction, and this leads to the aberration of light.

Problem : A source of light emitting light of wavelength 6563 Å is receding from an observer with a speed of 0.8c. Find the wavelength of light as observed by the observer.

What would be the observed wavelength if the source approaches the observer with the same velocity?

Given : $\lambda_0 = 6563\text{Å}$, $V = 0.8c$, $\lambda = ?$

Solution :

In Longitudinal Doppler effect, when the source and observer move away from each other,

$$\nu = \nu_0 \sqrt{\frac{c - V}{c + V}}$$

$$\therefore \frac{1}{\nu} = \frac{1}{\nu_0} \sqrt{\frac{c + V}{c - V}}$$

$$\therefore \frac{c}{\nu} = \frac{c}{\nu_0} \sqrt{\frac{c + V}{c - V}}$$

$$\therefore \lambda = \lambda_0 \sqrt{\frac{c+V}{c-V}}$$

$$= 6563 \sqrt{\frac{c+0.8\,c}{c-0.8\,c}} = 6563 \sqrt{\frac{1.8}{0.2}} = 19689 \text{ Å}$$

When the source and observer move towards each other,

$$\upsilon = \upsilon_0 \sqrt{\frac{c+V}{c-V}}$$

$$\therefore \frac{1}{\upsilon} = \frac{1}{\upsilon_0} \sqrt{\frac{c-V}{c+V}}$$

$$\therefore \frac{c}{\upsilon} = \frac{c}{\upsilon_0} \sqrt{\frac{c-V}{c+V}}$$

$$\therefore \lambda = \lambda_0 \sqrt{\frac{c-V}{c+V}}$$

$$= 6563 \sqrt{\frac{c-0.8\,c}{c+0.8\,c}} = 6563 \sqrt{\frac{0.2}{1.8}} = 2188 \text{ Å}$$

Problem : A source of light is emitting Sodium D_2 line (5890 Å) and is moving along a circle with constant speed 0.6c relative to an observer fixed at the centre of the circle. Find the wavelength of light as observed by the observer.

Given : $\lambda_0 = 5890$ Å, $V = 0.6\,c$, $\lambda = ?$

Solution :

For Transverse Doppler Effect,

$$\upsilon = \upsilon_0 \sqrt{1 - \frac{V^2}{c^2}}$$

$$\therefore \frac{1}{\upsilon} = \frac{1}{\upsilon_0} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore \frac{c}{\upsilon} = \frac{c}{\upsilon_0} \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore \lambda = \frac{\lambda_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{5890}{\sqrt{1-0.36}} = \frac{5890}{0.8} = 7363 \text{ Å.}$$

Problem : Show that the exact relativistic aberration formula

$$\tan\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{\cos\theta-\beta}$$

where $\beta = \frac{V}{c}$ can be derived from the Lorentz transformation equations of velocity.

Solution :

Let us consider a ray of light travelling with velocity c in X-Y plane in the direction making an angle θ with X - axis relative to the frame S as shown in Fig. 2.15.

$\therefore$ Component of velocity c along X - axis $= u_x = c\cos\theta$

Component of velocity c along Y - axis $= u_y = c\sin\theta$

The same ray is observed from the frame S' moving with uniform linear velocity V relative to the frame S along X - axis. In this frame, the ray travels with the same velocity c but its direction makes an angle θ' with X' - axis.

$\therefore$ Component of velocity c along X' - axis $= u'_x = c\cos\theta'$

Component of velocity c along Y'- axis $= u'_y = c\sin\theta'$

By Lorentz transformations of velocity,

$$u'_x = \frac{u_x - V}{1-\frac{Vu_x}{c^2}}, \qquad u'_y = \frac{u_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{Vu_x}{c^2}}$$

$$\therefore c\cos\theta' = \frac{c\cos\theta - V}{1-\frac{V.c\cos\theta}{c^2}}, \qquad c\sin\theta' = \frac{c\sin\theta\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V.c\cos\theta}{c^2}}$$

Dividing the second equation by the first equation,

$$\frac{c\sin\theta'}{c\cos\theta'} = \frac{c\sin\theta\sqrt{1-\frac{V^2}{c^2}}}{c\cos\theta - V}$$

$$\therefore \tan\theta' = \frac{c\sin\theta\sqrt{1-\frac{V^2}{c^2}}}{c\cos\theta - V} = \frac{\sin\theta\sqrt{1-\frac{V^2}{c^2}}}{\cos\theta-\frac{V}{c}}$$

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Now, $\frac{V}{c} = \beta$

$$\therefore \tan\theta' = \frac{\sin\theta\sqrt{1-\beta^2}}{\cos\theta - \beta}$$

This is nothing but the exact relativistic aberration formula. Thus it can also be derived from the Lorentz transformation equations of velocity.

2.23 A LIST OF IMPORTANT FORMULAE OF THE CHAPTER :

a) $x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}$, $y' = y$, $z' = z$. $t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$

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b) $\Delta x = (\Delta x)_0 \sqrt{1 - \frac{V^2}{c^2}}$

c) $\Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$

d) $u'_x = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}$ $u'_y = \frac{u_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$ $u'_z = \frac{u_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$

e) By Einstein's law of velocity addition,

$$U = \frac{V + W}{1 + \frac{V.W}{c^2}}$$

f) When the source and the observer are moving away from each other,

$$\upsilon = \upsilon_0\sqrt{\frac{c - V}{c + V}}, \quad \upsilon < \upsilon_0$$

g) When the source and the observer are moving towards each other,

$$\upsilon = \upsilon_0\sqrt{\frac{c + V}{c - V}}, \quad \upsilon > \upsilon_0$$

h) For transverse Doppler effect,

$$\upsilon = \upsilon_0\sqrt{1 - \frac{V^2}{c^2}}, \quad \upsilon < \upsilon_0$$

i) $\beta = \frac{V}{c}$

QUESTIONS

1. State the basic postulates of the Special theory of Relativity.
[B.U. Oct.1980, May 1981, April 1983, April 1987, April 1988, May 1990, Oct. 1990, April, 1992, May 1993, Oct. 1993]

2. State and explain the basic postulates of the Special theory of Relativity.
[B.U. April 1989, April 1991, Oct. 1992, May 1994]

3. Derive Lorentz transformation equations of space and time co-ordinates.
[B.U. Oct. 1980, May 1981, May 1982, Oct. 1982, April 1983, Oct. 1983, April 1986, Oct 1989, Oct. 1990, April 1991, Oct. 1991, April 1992, Oct. 1992, May 1993, Oct. 1993]

4. Show that Lorentz transformation equations reduce to Galilean transformation equations for small velocities.

5. Show that if two events are simultaneous according to an observer in one inertial frame, they are not necessarily simultaneous according to an observer in another inertial frame. [B.U. Oct. 1980]

6. Two events are occurring at the same point in one inertial frame but at different instants of time. Then show that they occur at different points in another inertial frame.

7. Show that the following wave equation of electrodynamics is invariant under Lorentz transformations.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

8. Using Lorentz transformation equations deduce the expressions for
   a) Length Contraction
   b) Time dilation.
[B.U. May 1980, Oct. 1980, Oct. 1982, Oct. 1983, Oct.1984, April 1985, April 1986, April 1987, April 1989, Oct. 1990, April 1991, Oct. 1991, Oct. 1992, May 1993]

9. Using Lorentz transformation equations derive velocity transformation equations.
[B.U. Oct. 1983, April 1984, Oct. 1987]

10. Two inertial frames of reference are moving with a relative velocity `V' along the common x - x' axis. If u_x and u'_x are the velocities of an object as measured in two frames parallel to the direction of their relative motion, then show that

$$u'_x = \frac{u_x - V}{1 - \frac{Vu_x}{c^2}}$$

Hence show that the velocity of light in free space has the same value 'c' in all inertial frames of reference. [B.U. April 1983]

11. Using Lorentz transformation equations derive the formula for relativistic addition of velocities. [B.U. May 1980, Oct. 1984, April 1988, Oct. 1990, Oct. 1991]

12. Using Lorentz transformation equations deduce the expressions for
   a) Relativistic addition of velocities along X - axis
   b) Transformation of velocities along Y - axis. [B.U. Oct. 1982, April 1992]

13. Show that Einstein's law of velocity addition leads to Fresnel's drag formula.
[B.U. Oct. 1987, Oct. 1989]

14. Derive the equation :

$$v' = \frac{c}{n} - u\left(1 - \frac{1}{n^2}\right),$$

where v' is speed of light moving through the liquid which is moving with a velocity u in the direction towards the source of light and n is the refractive index of the liquid. [B.U. April 1985].

15. Derive relativistic transformations for components of acceleration.

16. Derive the relativistic acceleration transformation equation :

$$a'_x = \frac{a_x (1 - \beta^2)^{3/2}}{\left(1 - \frac{u_x}{V}\beta^2\right)^3}.$$

where, $a_x = \frac{du_x}{dt}$, $a'_x = \frac{du'_x}{dt'}$, $\beta = \frac{V}{c}$

[B.U. April 1983, Oct. 1983 April 1987, April 1988, Oct. 1992, May 1993]

17. Considering the equation of a simple harmonic progressive wave travelling in X - Y plane derive relativistic expressions for
    a) Aberration of light.
    b) Longitudinal Doppler effect.
    c) Transverse Doppler effect.
18. a) Derive the relativistic equation for the aberration of light.
    b) Show that the observed first order aberration effect which corresponds to the classical picture, is a special case of the exact relativistic formula.

[B.U. Oct. 1987, April 1989]

19. Explain how the phenomenon of Stellar aberration can be explained by the relativity theory. [B.U. Oct. 1983]
    (Hint : Your answer should be the same as that of Q. No. 18)
20. Deduce the relativistic equations for Doppler Effect in light. [B.U. April 1983, Oct. 1992, Oct. 1993]
21. Write notes on the following :
    i) Relativity of simultaneity [B.U. May 1993]
    ii) Doppler effect in Relativity. [B.U. May 1990, May 1994]

PROBLEMS

1. At what relative velocity V will the Galilean and Lorentz transformations of x differ by (a) 1 percent (b) 10 percent of Lorentz transformations of x?
   [B.U. Oct. 1989] [(a) 0.1411 c (b) 0.4359 c].
2. A spaceship is 100m long on the ground. When it is in flight it appears to be 98 m long. Find its speed. Take $c = 3 \times 10^8$ m/s. [5.97×10^7 m/s].
3. An astronaut whose height is 1.8 m on the earth is lying parallel to the axis of the spaceship, in the direction of motion of the spaceship relative to the earth. If the velocity of the spaceship relative to the earth is 0.8 c find the height of the astronaut as measured by (i) an observer in the same spaceship (ii) an observer on the earth. [(i) 1.8 m (ii) 1.08 m].
4. A particle with proper life - time of 2×10^{-6} sec. , moves in a laboratory with a speed 0.9c. Calculate its life - time as measured in the laboratory.
   [4.588×10^{-6} sec.]
5. The life - time of a pion moving with velocity 0.8 c appears to be 2.5×10^{-7} sec. Calculate its proper life - time. [1.5×10^{-7} sec].
6. A rod is moving with velocity 0.6 c relative to an observer in the laboratory. Its length in the direction of motion appears to be 1 metre. What is its proper length? [1.25 m]
7. A spaceship is travelling at a speed of 0.8 c. Find the contraction of length observed as a fraction of its proper length. [0.4 or $\frac{2}{5}$]
8. The average life - time of a μ meson when measured at rest is 2.4×10^{-6} sec. What average distance would it travel before dying as measured in the reference frame in which its velocity is 0.8 c. Compare this distance with the distance which the meson sees travelling through by itself. Take c = 3×10^8 m/s.
   [960 m, 576 m].
9. A collimated beam of pions leaves the target of an accelerator at a speed of 0.8 c. The half life of pions is 1.77×10^{-8} sec. which is measured when the pions are brought to rest. Find at what distance from the target, the intensity of the beam will reduce to half. Take c = 3×10^8 m/s. [7.079 m]
10. The average life - time of muons stopped in a lead block in the laboratory is measured to be 2.3×10^{-6} sec. The average life - time of muons found in the bursts of cosmic rays observed from the earth is measured to be 4.6×10^{-6} sec. Find the speed of these cosmic ray muons. [0.866 c].
11. The average proper life - time of a μ meson is 2.3×10^{-6} sec. What average distance would it travel in vacuum before decaying as measured in the reference

frame in which its velocity is 0.9 *c*? Compare this distance with the distance the meson sees travelling through by itself. Take $c = 3 \times 10^8$ m/s. [B.U. April 1981]

[1424 m , 621 m]

12. What is the life of π mesons travelling with $\beta = 0.73$? What distance is travelled at $\beta = 0.73$ during one mean life? The proper mean life is 2.5×10^{-8} sec. Take $c = 3 \times 10^8$ m/s. [B.U. April 1986]

[Note : β stands for $\frac{V}{c}$ in general] [3.658×10^{-8} sec, 8.0121 m].

13. The mean life - time of μ mesons stopped in a lead block in the laboratory is measured to be 2.3×10^{-6} sec. The mean life of the high speed μ mesons in a burst of cosmic rays observed from the earth is measured to be 1.6×10^{-5} sec. Find the speed of cosmic ray μ mesons. [B.U. April 1984, Oct. 1987]

[0.9896 *c*]

14. The length of a spaceship is measured to be exactly half its proper length. (a) What is the speed of the spaceship relative to the observer? (b) What is the dilation of the spaceship's unit time? [B.U. April 1988, Oct. 1993]

[(a) 0.866 *c* (b) it is doubled]

15. Two spaceships each of proper length 100 m pass near each other heading in opposite directions. If an astronaut at the front of one ship measures a time - interval of 2.50×10^{-6} sec for the second ship to pass him, then (a) what is the relative velocity of the spaceships? (b) What time - interval is measured on the first ship for the front of the second ship to pass from the front to the back of the first ship? Take $c = 3 \times 10^8$ m/s. [B.U. Oct. 1982]

[(a) 3.965×10^7 m/s, (b) 2.523×10^{-6} sec.]

16. A circular loop having diameter 100 cm is at rest in X' - Y' plane of the frame S'. Find out its diameter in the direction of X - axis and Y - axis as observed by the observer in the frame S relative to which the frame S' is moving with velocity 0.8 *c* along X - axis. What is the shape of the loop for the observer in the frame S?

[60 cm, 100 cm, ellipse]

17. A spaceship, 1000 m long is moving with velocity 0.8*c* relative to you. Find the length of the ship as measured by you. How long will the ship take to pass you? $c = 3 \times 10^8$ m/s.

[600 m, 2.5×10^{-6} sec]

18. An observer in the laboratory observes that an atom moving with a velocity 2×10^8 m/s emits a β particle moving in the same direction with velocity 2.8×10^8 m/s relative to the atom. Find the velocity of the β - particle relative to the observer. Given $c = 3 \times 10^8$ m/s.

[2.958×10^8 m/s]

19. Two electrons leave a radioactive sample in opposite directions with velocities 0.8 *c* and 0.9 *c* respectively relative to the sample. Find velocity of one electron relative to the other electron.

[0.9884 *c*].

20. An observer is moving with a velocity 2.7×10^8 m/s relative to a frame of reference attached to a laboratory. He finds that a particle is approaching him with a velocity 2.998×10^8 m/s. What is the velocity of the particle relative to the laboratory frame? Take $c = 3 \times 10^8$ m/s.

[-2.963×10^8 m/s]

21. Two electrons approach each other with velocities 0.8 c and 0.6 c along the same line relative to an observer. Calculate their relative velocity.

[-0.9459 *c*]

[B.U. April 1980]

22. Two electrons move away from each other with the same velocity 0.9*c* along the same line relative to an observer. Calculate their relative velocity. [0.9945 *c*].

23. An electron is moving with a speed 0.8 *c* in a laboratory frame. A photon is moving with a velocity *c* in the opposite direction in the laboratory frame. Find the velocity of the photon relative to the electron. [-*c*]

24. An observer finds that two photons are approaching each other with the same velocity c. Find the velocity of one photon relative to the other photon. [-*c*]

25. Two spaceships are observed from the earth to be approaching each other with a velocity of 0.5 c each. With what velocity is each spaceship approaching the other as seen from either ship? [B.U. Oct. 1983, May 1993]
[-0.8 c]

26. Two spaceships are observed from the earth to be approaching each other with a velocity of 0.6 c each. With what velocity is each spaceship approaching the other as seen from either ship? [B.U. April 1987]
[- 0.8824 c.]

27. A source of light emitting light of wavelength 5893 Å is receding from an observer with a speed of 0.5 c. Find the wavelength of light as observed by the observer. [10220 Å]

28. A source of light emitting light of wavelength 6000 Å is approaching an observer with a speed of 0.8 c. Find the wavelength of light as observed by the observer [2000 Å]

29. Find the shift in wavelength due to longitudinal Doppler effect for sodium D_1 line (5896 Å) when the source and the observer are approaching each other at a relative velocity 0.6 c. [2948 Å]

30. Find the shift in wavelength in Doppler effect for light of wavelength 6000 Å emitted by a source moving along a circle with speed 0.8 c relative to an observer fixed at the centre of the circle. [4000 Å]

31. A spaceship is receding from the earth at a velocity V. Wavelength $\lambda = 1216$Å of ultraviolet rays on the earth appears to be $\lambda = 3648$ Å to the observer in the spaceship. Find the velocity V of the spaceship relative to the earth. [$V = 0.8\,c$]

FOOD FOR THOUGHT

1. Thus the theory is, in a sense, profoundly conservative; the principles that have been found to hold for dynamics are stated to hold for the whole of physics. The special theory of Relativity is a necessary consequence of any assertion that the unity of physics is essential, for it would be intolerable for all inertial systems to be equivalent from a dynamical point of view yet distinguishable by optical measurements. It is now almost incredible that the possibility of such a discrimination was taken for granted in nineteenth century.

- Hermann Bondi

* * * * *

2. The Relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties, using only a few very convincing assumptions.... The old mechanics is valid for small velocities and forms the limiting case of the new one.

[Evolution of Physics]

- Albert Einstein & Leopold Infeld

* * * * *

3. The illusion which prevailed prior to the enunciation of the Theory of Relativity - that, from the point of view of experience the meaning of simultaneity in relation to spatially distant events and, consequently, that the meaning of physical time is a priori clear - this illusion had its origin in the fact that in our everyday experience we can neglect the time of propagation of light. We are accustomed on this account to fail to differentiate between "simultaneously seen" and "simultaneously occurring", and, as a result, the difference between time and local time is blurred.

- Albert Einstein

* * * * *

4. We have to take into account that all our judgements in which time plays a part are always judgements of simultaneous events. If, for instance, I say "That train arrives here at 7 o'clock", I mean something like this: "The pointing of the small hand of my watch to 7 and the arrival of the train are simultaneous events."

- Albert Einstein

* * * * *

5. **What is a clock?**

The primitive subjective feeling of time flow enables us to order our impressions, to judge that one event takes place earlier, another later. But to show that the time interval between two events is 10 seconds a clock is needed. By the use of a clock the time concept becomes objective. Any physical phenomenon may be used as a clock, provided it can be exactly repeated as many times as desired. Taking the interval between the beginning and the end of such an event as one unit of time, arbitrary time intervals may be measured by repetition of this physical process. All clocks, from the simple hour-glass to the most refined instruments, are based on this idea. With the hour-glass the unit of time is the interval the sand takes to flow from the upper to the lower glass. The same physical process can be repeated by inverting the glass.

- Albert Einstein & Leopold Infeld

[Evolution of Physics]

* * * * *

6. To use a comparison, we could say that creating a new theory is not like destroying an old barn and erecting a skyscraper in its place. It is rather like climbing a mountain, gaining new and wider views, discovering unexpected connections between our starting point and its rich environment. But the point from which we started out still exists and can be seen, although it appears smaller and forms a tiny part of our broad view gained by the mastery of the obstacles on our adventurous way up.

- Albert Einstein & Leopold Infeld

(Evolution of Physics)

* * * * *

7. Let us consider a man eating his dinner in the dining car of a moving train. From the point of views of the dining - car waiter,he eats his appetizer and his dessert at the very same place (third table near the window). But from the point of view of two switchmen at stationary points on the railroad track looking through the window of the car the two events take place many miles apart. Thus we may say that : *Two events occurring at the same place, but at two different moments from the point of view of one observer will be considered as occurring at different places if viewed by other observers in a different state, or in different states of motion.*

In view of the desired space - time equivalence, replace in the above sentence the word "place" by the word "moment" and vice versa. The sentence will now read : *The two events occurring at the same moment, but at different places, from the point of view of one observer will be considered as occurring at different moments if viewed by another observer in a different state of motion.*

In application to our dining - car example, we would expect that whereas the waiter would swear that two passengers sitting at opposite ends of the car lighted their after - dinner cigarettes at exactly the same moment, a switchman standing still on the track and looking through the windows as the train moves part him would insist that one of these gentlemen had done it before the other.

Thus : Two events considered to be simultaneous from the point of view of one observer will from the point of view of another observer be separated by a certain time interval.

[One, Two, Three ... Infinity] *- George Gamow*

* * * * *

8. If you consider the matter further you will also easily realize that each physical object has four dimensions, three in space and one in time. Thus the house in which you live extends so much in length, width, height and time, the last extension being measured by the period of time from the date that house was built to the date it will finally burn down, or be taken apart by some wrecking company or disintegrate at the end of an advanced old age.

- George Gamow

* * * * *

9. From this we conclude that in the Theory of Relativity the velocity c plays the part of a limiting velocity, which can neither be reached nor exceeded by any real body. ..... According to the Theory of Relativity, the action at a distance with the velocity of light always takes the place of instantaneous action at a distance or of action at a distance with an infinite velocity of transmission.

- Albert Einstein

* * * * *

10. Special theory of Relativity has shown generally the role which the universal constant *c* (velocity of light) plays in the laws of nature and has demonstrated that there exists a close connection between the form in which time on the one hand and the spatial co-ordinates on the other hand enter into the laws of nature.

- *Albert Einstein*

* * * * *

11. We can regularly obtain particles of velocities approaching that of light in vacuum, particles in regard to which it is necessary to take account of corrections introduced by the Special theory of Relativity.

- *Louis de Broglie*

* * * * *

12. The special theory of Relativity has led to a clear understanding of the physical concepts of space and time and in connection with this to a recognition of the behaviour of moving measuring rods and clocks. It has in principle removed the concept of absolute simultaneity and thereby also that of instantaneous action at a distance in the sense of Newton. It has shown how the laws of motion must be modified in dealing with velocities that are not negligibly small as compared with the velocity of light.

- *Albert Einstein*

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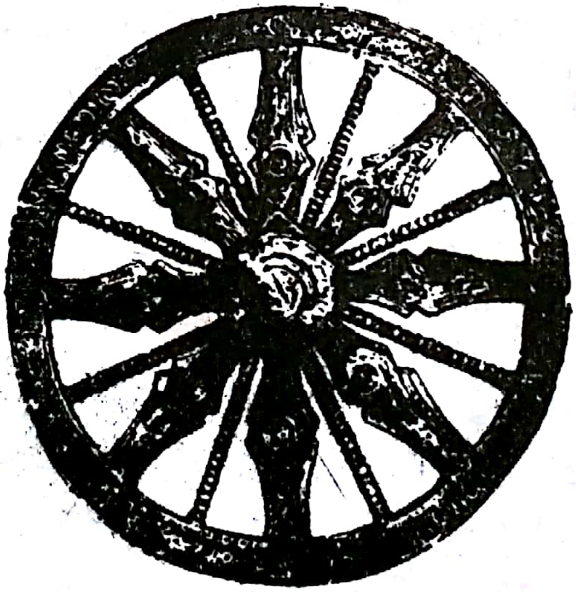


3

Relativistic Mechanics

In this chapter we shall develop Relativistic Mechanics. We have seen in the first chapter that the laws of Mechanics such as Newton's laws of motion, the principle of conservation of momentum etc. are invariant in all inertial frames of reference under Galilean transformations.

In the second chapter we learned that Lorentz transformation equations are the correct transformation equations for invertial frames of reference and they should replace the Galilean transformation equations. According to the first postulate of the Special theory of Relativity all the laws of Physics including the laws of Mechanics and Electrodynamics should be invariant in all inertial frames of reference under Lorentz transformations.

Now, the laws of Electrodynamics are invariant under Lorentz transformations as we have observed in the second chapter, but the laws of Mechanics are not invariant under Lorentz transformations. In the present chapter we shall strive to make the laws of Mechanics invariant under Lorentz transformations. For this purpose we shall have to generalize them, modify them from their classical form. This modification should be such that they reduce to their classical form for small velocities as compared to the velocity of light c.

First we shall take up the principle of conservation of momentum. In order to make it invariant under Lorentz transformations, as you will see, we shall have to give up the absolutism of mass i.e. constancy of mass. (In Classical Mechanics mass of a body remains constant eternally for all observers). We have already accepted the relativity of simultaneity, space and time in the second chapter. Now we shall have to accept the relativity of mass in trying to make the principle of conservation of momentum invariant. In the process we shall get an expression for the variation of mass with velocity which would be analogous to the expressions for the variation of length and time-interval with velocity.

3.1 TO DERIVE AN EXPRESSION FOR THE VARIATION OF MASS WITH VELOCITY :

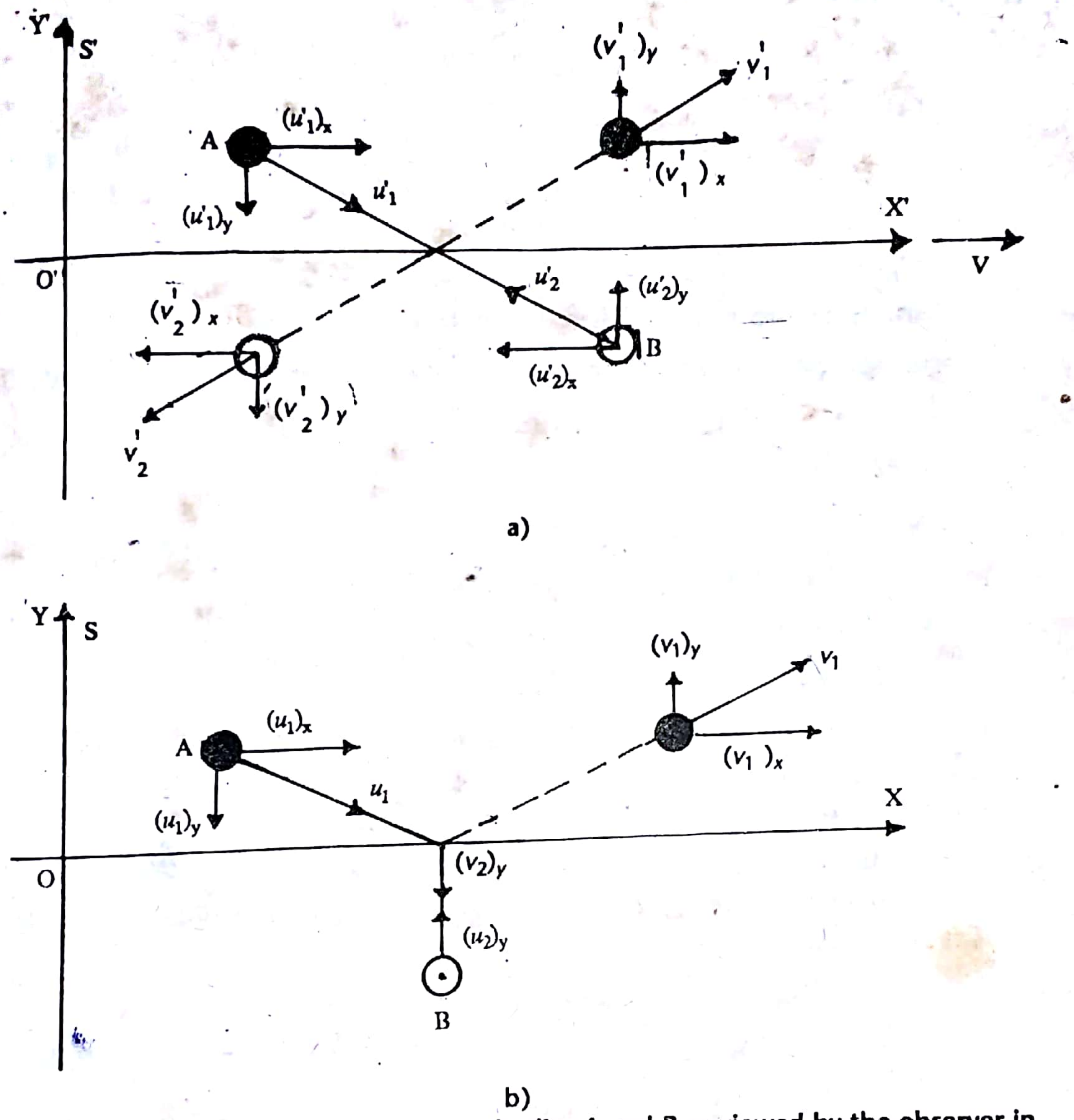


Fig. 3.1 (a) Elastic collision between two bodies A and B as viewed by the observer in the frame S'. (b) The same collision as viewed by the observer in the frame S.

Let us consider two exactly identical bodies A and B, each having mass m in the frame S'. [Fig. 3.1 (a)]. They are moving in X'-Y' plane towards each other with velocities which are equal in magnitude but opposite in direction.

$$\therefore\ (u'_1)_x = -(u'_2)_x, \qquad (u'_1)_y = -(u'_2)_y \qquad \ldots(3.1)$$

The bodies undergo a perfectly elastic collision. The collision is highly symmetric for the observer in the frame S'. The total initial momentum of the system is zero. By the principle of conservation of momentum, the total final momentum of the system must be zero. Hence the final velocities of A and B also are equal in magnitude, opposite in direction [Fig. 3.1 (a)]. Moreover, since the collision is perfectly elastic the total kinetic energy of the system remains the same. Hence the final velocities of A and B have the same magnitude as their respective initial velocities.

i.e. $(v'_1)_x = (u'_1)_x \qquad (v'_2)_x = (u'_2)_x$

$(v'_1)_y = -(u'_1)_y \qquad (v'_2)_y = -(u'_2)_y$ ...(3.2)

Thus due to the collision, only the directions of the Y-components of the velocities of A and B are reversed whereas their X-components remain the same.

Applying the principle of conservation of momentum in Y-direction, since no external force is acting on the system,

momentum gained by the body A=momentum lost by the body B.

$$m\,(v'_1)_y - m\,(u'_1)_y = m\,(u'_2)_y - m\,(v'_2)_y$$

By equations (3.2),

$$(v'_1)_y = -(u'_1)_y, \quad (v'_2)_y = -(u'_2)_y.$$

Hence the above equation becomes

$$-m\,(u'_1)_y - m\,(u'_1)_y = m\,(u'_2)_y + m\,(u'_2)_y$$

$$\therefore -2m\,(u'_1)_y = 2m\,(u'_2)_y$$

i.e. $m\,(u'_1)_y = -m\,(u'_2)_y$ ...(3.3)

This equation represents the principle of conservation of momentum in Y-direction relative to the frame S'.

Let the same collision be observed by an observer in the frame S relative to which the frame S' is moving with uniform linear velocity V along X-axis.

Let us select V in such a way that

$V = -(u'_2)_x$ ...(3.4)

Hence the body B is **stationary** relative to the frame S in X-direction before the collision and even after the collision as shown in Fig. 3.1 (b) i.e. $(u_2)_x = 0$ and $(v_2)_x = 0$. It has velocity only in Y-direction before and after the collision.

Now we shall transform the equation (3.3) to the frame S. By Lorentz transformations of velocity,

$$u'_y = \frac{u_y\sqrt{1-\dfrac{V^2}{c^2}}}{1-\dfrac{V\,.\,u_x}{c^2}}$$

$$\therefore (u'_1)_Y = \frac{(u_1)_Y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V(u_1)_x}{c^2}} \qquad \text{...(3.5)}$$

$$(u'_2)_Y = \frac{(u_2)_Y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V(u_2)_x}{c^2}}$$

But $(u_2)_x = 0$

$$\therefore (u'_2)_Y = (u_2)_Y\sqrt{1 - \frac{V^2}{c^2}} \qquad \text{...(3.6)}$$

By equations (3.5) and (3.6), equation (3.3) becomes,

$$m\frac{(u_1)_Y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V(u_1)_x}{c^2}} = -\,m\,(u_2)_Y\sqrt{1 - \frac{V^2}{c^2}}$$

$$\therefore \frac{m\,(u_1)_Y}{1 - \frac{V(u_1)_x}{c^2}} = -\,m\,(u_2)_Y \qquad \text{...(3.7)}$$

Equations (3.3) and (3.7) represent the principle of conservation of momentum in the frame *S'* and *S* respectively. According to the first postulate of the Special theory of Relativity they should have the same form. But they do not have the same form.

In order to make the form of equation (3.7) same as that of equation (3.3) we assume that the two bodies *A* and *B* do not have the same mass in the frame S. Let their masses relative to the frame S be m_1 and m_2 respectively. According to the first postulate of the Special theory of Relativity, the equation representing the principle of conservation of momentum in Y - direction relative to the frame S must have the same form as the equation (3.3)

i.e. $m_1\,(u_1)_Y = -\,m_2\,(u_2)_Y$.

$$\therefore \frac{m_1}{m_2} = -\,\frac{(u_2)_Y}{(u_1)_Y} \qquad \text{...(3.8)}$$

Taking equation (3.1)

i.e. $(u'_1)_Y = -\,(u'_2)_Y$.

By equation (3.5) and (3.6), this equation becomes,

$$\frac{(u_1)_Y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V(u_1)_x}{c^2}} = -\,(u_2)_Y\sqrt{1 - \frac{V^2}{c^2}}$$

$$\therefore -\frac{(u_2)_y}{(u_1)_y} = \frac{1}{1-\frac{V(u_1)_x}{c^2}}$$

Hence equation (3.8) becomes,

$$\frac{m_1}{m_2} = \frac{1}{1-\frac{V(u_1)_x}{c^2}}$$

$$\therefore m_1 = \frac{m_2}{1-\frac{V(u_1)_x}{c^2}} \qquad \text{...(3.}$$

Now, we shall express V in terms of $(u_1)_x$ as follows :

By equation (3.4),

$$V = -(u'_2)_x = (u'_1)_x \text{ according to equation (3.1).}$$

$$V = \frac{(u_1)_x - V}{1-\frac{V(u_1)_x}{c^2}}$$

$$\therefore V - \frac{V^2(u_1)_x}{c^2} = (u_1)_x - V$$

$$\therefore \frac{(u_1)_x}{c^2} \cdot V^2 - 2V + (u_1)_x = 0$$

$$\therefore (u_1)_x V^2 - 2c^2 V + (u_1)_x c^2 = 0$$

This is a quadratic equation in V.

Form of the equation : $ax^2 + bx + c = 0$.

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore V = \frac{+2c^2 \pm \sqrt{4c^4 - 4(u_1)_x^2 c^2}}{2(u_1)_x}$$

$$= \frac{2c^2 \pm 2c^2\sqrt{1-\frac{(u_1)_x^2}{c^2}}}{2(u_1)_x}$$

$$= \frac{c^2 \pm c^2\sqrt{1-\frac{(u_1)_x^2}{c^2}}}{(u_1)_x}$$

Now, V cannot be greater than c. Hence we have to select negative sign on R.H.S.

$$\therefore V = \frac{c^2 - c^2\sqrt{1 - \frac{(u_1)_x^2}{c^2}}}{(u_1)_x}$$

$$= \frac{c^2\left(1 - \sqrt{1 - \frac{(u_1)_x^2}{c^2}}\right)}{(u_1)_x}$$

Substituting this expression for V in equation (3.9) we get,

$$m_1 = \frac{m_2}{1 - \frac{c^2\left(1 - \sqrt{1 - \frac{(u_1)_x^2}{c^2}}\right)}{(u_1)_x} \cdot \frac{(u_1)_x}{c^2}}$$

$$= \frac{m_2}{1 - \left(1 - \sqrt{1 - \frac{(u_1)_x^2}{c^2}}\right)}$$

$$= \frac{m_2}{\sqrt{1 - \frac{(u_1)_x^2}{c^2}}}$$

Since this equation contains only X - component of velocity we shall now consider the motion in X - direction only. Then the body B is stationary relative to the frame S. Hence its mass $m_2 = m_0$ called its proper mass or rest mass.

$$\therefore m_1 = \frac{m_0}{\sqrt{1 - \frac{(u_1)_x^2}{c^2}}}$$

[The body A is moving with velocity $(u_1)_x$. Hence its mass m_1 is varying with respect to its velocity $(u_1)_x$.]

If $(u_1)_x = 0$, then $m_1 = m_0$. i.e. if the body A also is stationary in the frame S, its mass equals its rest mass or proper mass which would be the same as that of B. When it is moving with a velocity u its mass is given by

$$\boxed{m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}}$$

This is the expression giving the variation of mass with velocity.

Remarks :

i) As the velocity of a particle u increases, the denominator $\sqrt{1-\frac{u^2}{c^2}}$ decreases and the mass of the particle m increases i.e., as the velocity of a particle increases its mass goes on increasing. Hence the proper mass or rest mass is the smallest mass of the particle. Mass is not absolute but a relative quantity.

ii) As $u \to c$, $m \to \infty$. Hence a particle can not attain the velocity of light. This happens because as the velocity of the particle increases its mass also increases. Hence its inertia increases and it becomes more and more difficult to accelerate the particle upto the velocity $c = 3 \times 10^8$ m/s.

iii) If $u > c$ then $\frac{u^2}{c^2} > 1$, $1 - \frac{u^2}{c^2}$ become negative and $\sqrt{1-\frac{u^2}{c^2}}$ becomes imaginary. But then m also becomes imaginary. This is not possible in nature. Hence a particle cannot move with a velocity u greater than the velocity of light c.

Thus a real body can neither attain the velocity of light nor travel faster than light. Once again we come to the conclusion that the velocity of light $c = 3 \times 10^8$ m/s in empty space is the limiting velocity in the universe.

iv) The formula, $m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$ has been verified experimentally for different beams of electrons moving with different velocities u, comparable with the velocity of light c. For each beam, the ratio e/m of the charge to mass of the electrons is determined in usual experiments involving the deflection of the electron - beam under the action of electric and magnetic fields. It has been observed that the mass of the electron varies with its velocity and the variation occurs according to the above formula.

3.2 RELATIVISTIC DEFINITIONS OF MOMENTUM AND FORCE :

Relativistic Momentum : If a particle is moving with a velocity $\vec{u}$ then its relativistic momentum is defined as

$$\vec{p} = m \cdot \vec{u} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}$$

where m_0 is the rest mass of the particle. Its components along X, Y, and Z-axes are as follows :

$$p_x = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\,u_x \;,\; p_y = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\,u_y \;,\; p_z = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\,u_z$$

Relativistic force :

By Newton's Second Law of motion,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$= \frac{d}{dt}\left[m\,.\,\vec{u}\right]$$

$$= \frac{d}{dt}\left[\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\,\vec{u}\right]$$

This is the relativistic formula for force.

Note : In Classical Mechanics, m is constant and hence

$$\vec{F} = m\frac{d\vec{u}}{dt} = m\,.\,\vec{a}$$

Thus $\vec{F} = m\,.\,\vec{a}$ is the formula for force in Classical Mechanics but not in Relativistic Mechanics.

3.3 MASS-ENERGY RELATION :

Let us consider the formula,

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

Since $\frac{u^2}{c^2}$ is always positive, $\sqrt{1-\frac{u^2}{c^2}}$ is always less than 1 and hence m is always greater than m_0 i.e. when a particle is in motion its mass becomes greater than its rest mass. For example a moving cricket ball has more mass than the same ball when at rest. Now the question is "From where does this extra mass come?" It cannot come out of nothing! The difference between a particle at rest and the particle in motion is that the moving particle has kinetic energy. This kinetic energy must be appearing as mass due to which the mass of the moving particle becomes greater than its rest mass. This consideration leads us to conclude that there must be some connection between the energy and mass. In what follows, we shall obtain this connecting relation between the energy and mass.

Derivation :

Let us consider a free particle which is stationary relative to an observer. Let its mass be m_0 .

A force F directed along X - axis acts on the particle due to which the particle acquires a velocity u along X - axis relative to the same observer. Let its corresponding mass be m so that

$$m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Work done by the external force $= \int F\, dx$

Since the particle is a free particle this work is converted into the kinetic energy of the particle. Moreover, since the particle was at rest initially its initial kinetic energy is zero. Hence the final kinetic energy of the particle equals the work done by the external force.

$$\text{i.e. } E_k = \int F\, dx$$

$$= \int \frac{dp}{dt}\, dx$$

$$= \int d\,(mu)\, .\, \frac{dx}{dt}$$

$$= \int (m\, du + u\, dm)\, .\, u$$

$$= \int mu\, du + \int u^2 . dm \qquad \text{...(3.10)}$$

$$\text{Now, } m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\therefore \frac{dm}{du} = \frac{d}{du}\left[\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}\right]$$

$$= m_0 \frac{d}{du}\left(1 - \frac{u^2}{c^2}\right)^{-1/2}$$

$$= m_0 \left[-\frac{1}{2}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\left(0 - \frac{2u}{c^2}\right)\right]$$

$$= m_0 \left[\frac{u}{c^2}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\right]$$

$$= \frac{m_0 u}{c^2\left(1 - \frac{u^2}{c^2}\right)^{3/2}}$$

$$= \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \cdot \frac{u}{c^2\left(1 - \frac{u^2}{c^2}\right)} = \frac{mu}{c^2 - u^2}$$ ✓

$$\therefore mu\, du = (c^2 - u^2)\, dm$$

$\therefore$ Equation (3.10) becomes,

$$E_k = \int (c^2 - u^2)\, dm + \int u^2\, dm$$

$$= \int_{m_0}^{m} (c^2 - u^2 + u^2)\, dm$$

$$= \int_{m_0}^{m} c^2\, dm$$

$$= c^2 \int_{m_0}^{m} dm \quad (\because c \text{ is constant})$$

$$= c^2 (m - m_0) \qquad \text{.....(3.11)}$$

$$\therefore \frac{E_k}{c^2} = m - m_0$$ ✓

This equation shows that the kinetic energy of the particle can be expressed as the increase in the mass of the particle over its rest mass. i.e. energy can be expressed in terms of mass on dividing the energy by c^2.

Conversely, $(m - m_0)\, c^2 = E_k$

This equation shows that mass can be expressed in terms of energy on multiplying the mass by c^2.

Hence the rest mass m_0 of the particle also must be associated with an amount of energy m_0c^2 called its **rest energy**.

Now, when the particle is in motion its total energy E is the sum of its rest energy and its kinetic energy.

i.e. $\boxed{E = m_0c^2 + E_k}$

$$= m_0c^2 + c^2(m - m_0) \text{ by equation (3.11)}$$

$$= m_0c^2 + mc^2 - m_0c^2$$

$$= mc^2$$

Thus the mass-energy relation is

$$\boxed{E = mc^2}$$

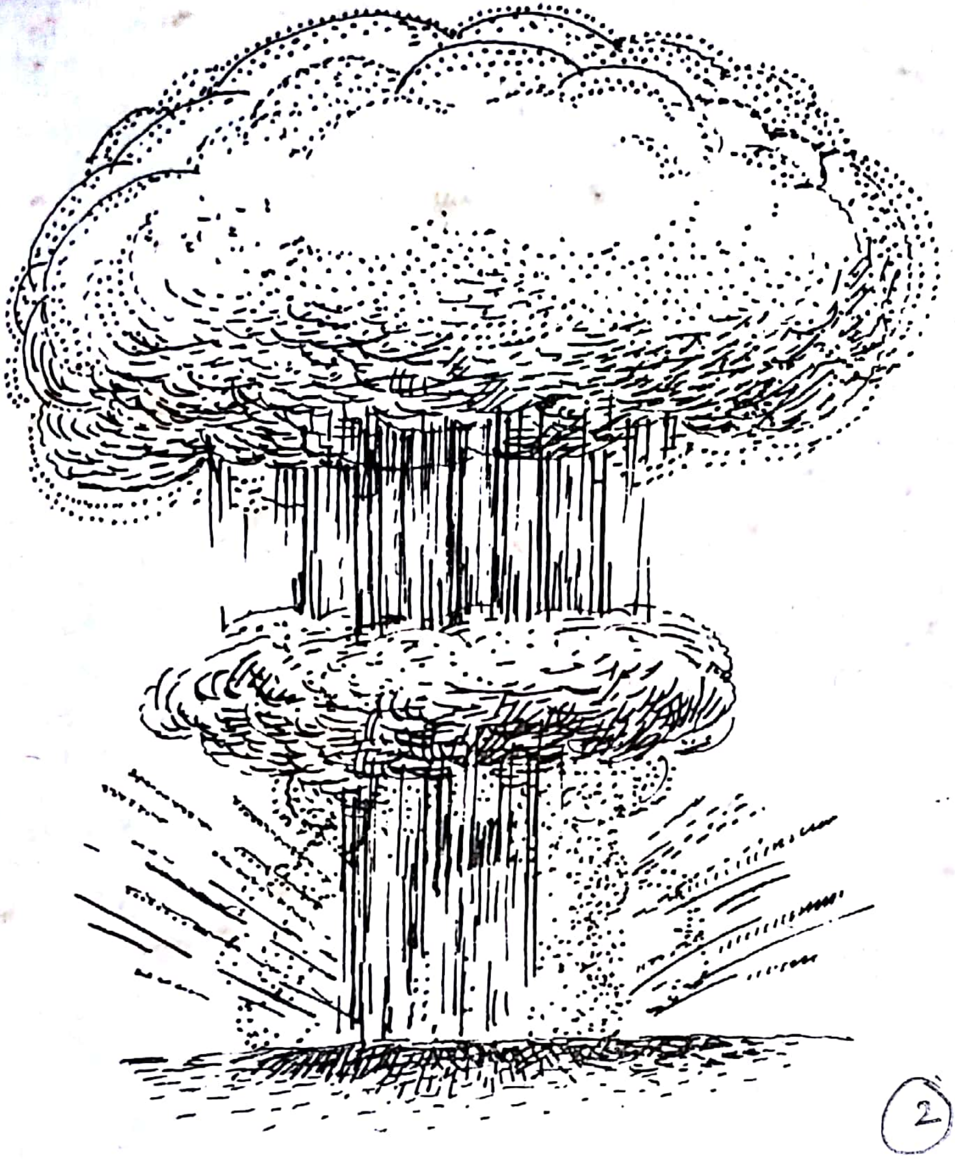
Discussion : ①

i) The equation, $E = mc^2$ is one of the most famous equations of Physics. It was originally derived by Einstein who considered it to be the most significant consequence of his Special theory of Relativity.

ii) The equation, $E = mc^2$ shows that mass and energy are equivalent quantities. Mass m is equivalent to energy $E = mc^2$. Conversely, energy E is equivalent to mass $m = \frac{E}{c^2}$. Hence it is possible to express mass in the units of energy and energy in the units of mass. There is no essential distinction between mass and energy. Energy has mass and mass represents energy.

iii) The factor $c^2 = 9 \times 10^{16}$ m²/s² is tremendously large. Hence a very small mass m becomes equivalent to an immense amount of energy $E = mc^2$. For example, a **mass of 1 gm** is sufficient to provide energy for the domestic purpose in a city of one million people for one day! In fact the conversion of mass into energy is the source of energy of the sun and the stars, atomic and hydrogen bombs and our atomic power stations. The sun and the stars are losing mass continuously due to the conversion of mass into energy which is emitted in the form of radiation.

iv) The mass-energy relation indicates the possibility of the conversion of energy into mass according to the equation $m = \frac{E}{c^2}$. Again since c^2 is tremendously large the mass m equivalent to energy E has an extremely small value. The energy can be in any form such as kinetic energy, heat energy, electrical energy and so on. For example a moving cricket ball has more mass than same ball when at rest. A heated piece of iron has more mass than the same piece when cold. But the excess mass in every case is too small to be detected.



②

Fig. 3.2 Conversion of mass into energy in atomic and hydrogen bombs

In fact, this mass which is equivalent to different forms of energy has all usual properties of mass such as inertia, weight, discreteness etc.

If an adequately large amount of energy is made available in concentrated form then practically it can be converted into mass and new particles can be created. In 1932, C. D Anderson found that actually two particles, an electron and a positron, could be produced from the energy of a gamma ray photon.

Conversely, it has been experimentally observed that an electron and a positron annihilate each other and produce gamma ray photons. i.e. mass can be converted into energy.

v) It is observed in Nuclear Physics that mass of a nucleus is slightly less than the total mass of the particles forming the nucleus. The loss of mass is called **mass defect**. The energy equivalent to mass defect is called **binding energy**. This energy is

③ responsible for binding the nucleons together into the nucleus. It is a measure of the stability of the nucleus i.e. the higher the binding energy the greater is the stability of the nucleus. If we want to break up the nucleus into the constituent particles we have to supply an amount of energy equal to binding energy from outside.

vi) In Classical Mechanics we have two separate principles of conservation viz. (a) Principle of conservation of mass and (b) Principle of conservation of energy. In Special theory of Relativity there is no difference between mass and energy. Energy has mass and mass represents energy. Mass and energy are equivalent and one can be expressed in terms of the other. Hence in Relativity the two principles of conservation merge into one conservation principle, namely, the principle of conservation of mass -energy.

Problem : A body of rest mass m_0 travelling initially at a speed of 0.6c makes a completely inelastic collision with an identical body initially at rest. What is the speed of the resulting single body?

Solution :

When the first body is moving with velocity 0.6 c its mass is given by

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{m_0}{\sqrt{1-0.36}} = \frac{m_0}{0.8}$$

$$\therefore \text{Initial momentum of the first body} = mu = \frac{m_0}{0.8} \times 0.6c = \frac{3m_0c}{4}$$

$$\text{Initial momentum of the second body} = 0$$

$$\therefore \text{Total initial momentum of the system} = p_1 = \frac{3m_0c}{4}$$

Let the velocity of the resulting single body be v. Let its corresponding mass be M.

$$\therefore \text{Final momentum of the system} \quad p_1 = Mv.$$

By the Principle of conservation of momentum, since no external force is acting on the system,

$$p_1 = p_2$$

$$\text{i.e.} \quad \frac{3m_0c}{4} = Mv \qquad \ldots (a)$$

$$\text{Total energy of the first body} = mc^2 = \frac{m_0c^2}{0.8}$$

$$\text{Total energy of the second body} = m_0c^2$$

$\therefore$ Total initial energy of the system $= E_1 = \dfrac{m_0c^2}{0.8} + m_0c^2$

Total final energy of the system $= E_2 = Mc^2$

By Principle of conservation of energy,

$$E_1 = E_2$$

$$\frac{m_0c^2}{0.8} + m_0c^2 = Mc^2$$

$$\therefore M = m_0\left[\frac{1}{0.8} + 1\right] = m_0\left[\frac{1.8}{0.8}\right] = \frac{9\,m_0}{4}.$$

$\therefore$ Equation (a) becomes

$$\frac{3\,m_0c}{4} = \frac{9\,m_0}{4}\,v$$

$$\therefore v = \frac{c}{3}$$

$\therefore$ Velocity of the resulting single body $= v = \dfrac{c}{3}$

Note : You must have noticed in the above problem that mass of the resulting single body comes out to be

$$M = \frac{9\,m_0}{4} = 2.25\,m_0\,.$$

By principle of conservation of mass of Classical Mechanics this mass would have been,

$$M = m_0 + m_0 = 2\,m_0.$$

In Relativistic Mechanics there is no separate conservation principle for mass and for energy. They merge into one conservation principle of mass - energy.

3.4 PHOTONS AS PARTICLES WITH ZERO REST MASS :

According to Quantum theory, a photon of frequency ν has energy given by

$\epsilon = h\nu$ where h is Planck's constant. According to the Special theory of Relativity, this energy is equivalent to mass m such that

$$\epsilon = mc^2$$

$$\therefore m = \frac{\epsilon}{c^2} = \frac{h\nu}{c^2}$$

$\therefore$ Mass of a photon $= m = \dfrac{h\nu}{c^2}$

A photon always travels with velocity c in empty space. Hence, momentum of the photon is given by

$$p = m \cdot c = \frac{h\nu}{c^2} \cdot c = \frac{h\nu}{c}$$

Moreover, $m = \dfrac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$

$$\therefore m_0 = m\sqrt{1 - \frac{u^2}{c^2}}$$

Now, as we have seen, a photon travels with velocity c in empty space and it has a finite mass i.e. for a photon, $u = c$ and m is finite.

$$\therefore m_0 = m\sqrt{1 - \frac{c^2}{c^2}} = m \times 0 = 0$$

Thus the rest mass of a photon is zero.

According to the second postulate of the Special theory of Relativity, the velocity of light in empty space has one and the same value $c = 3 \times 10^8$ m/s relative to all inertial frames of reference. Hence the photons travel in empty space with one and only one velocity $c = 3 \times 10^8$ m/s relative to all inertial frames of reference. Actually there is no real inertial frame in which a photon would be at rest. Thus the rest mass of photons is an unobserved quantity.

3.5 RELATIVISTIC KINETIC ENERGY :

In section 3.3, we have derived the equation (3.11) giving relativistic kinetic energy,

$$\text{i.e. } E_k = c^2(m - m_0)$$

$$= c^2\left[\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} - m_0\right]$$

$$= m_0c^2\left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1\right]$$

Thus the expression for the relativistic kinetic energy is

$$\boxed{E_k = m_0c^2\left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1\right]}$$

Note :

i) In Relativity, we should always use the above equation for kinetic energy. The following equation

$$E_k = \frac{1}{2} mu^2 = \frac{1}{2} \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \cdot u^2$$

which is obtained using the ideas from Classical Mechanics and Relativistic Mechanics is no longer true in Relativity.

ii) As $u \rightarrow c$, $E_k \rightarrow \infty$ which is not possible. Hence a particle cannot attain the velocity of light.

Moreover, if $u > c$ then $\frac{u^2}{c^2} > 1$, $1 - \frac{u^2}{c^2}$ becomes negative and $\sqrt{1 - \frac{u^2}{c^2}}$ becomes imaginary. But then E_k also becomes imaginary. This is not possible in nature. Hence a particle cannot move with a velocity u greater than the velocity of light c.

Thus a real body can neither attain the velocity of light nor travel faster than light. The velocity of light $c = 3 \times 10^8$ m/s in empty space is the limiting velocity in the universe.

iii) In the following, we shall investigate whether the Relativistic expression for Kinetic energy reduces to the Newtonian expression for kinetic energy for small velocities.

The Relativistic expression for kinetic energy is

$$E_k = m_0c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right]$$

$$= m_0c^2 \left[\left(1 - \frac{u^2}{c^2}\right)^{-1/2} - 1 \right] \qquad \text{...(3.12)}$$

By Binomial Theorem,

$$(a + x)^n = a^n + na^{n-1} . x + \frac{n(n-1)}{2} a^{n-2} . x^2 + ...$$

$$\therefore \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = (1)^{-1/2} + \left(-\frac{1}{2}\right) (1)^{-3/2} \left(-\frac{u^2}{c^2}\right) + \text{terms in higher powers of } \frac{u^2}{c^2}$$

$$= 1 + \frac{1}{2} \frac{u^2}{c^2} + \text{terms in higher powers of } \frac{u^2}{c^2}$$

In Classical Mechanics we deal with the particles having velocity u much less than the velocity of light.

i.e. $u << c \quad \therefore \frac{u}{c} << 1.$

Hence the terms in higher powers of $\frac{u}{c}$ becomes negligible in comparison with the first two terms on R.H.S.

$$\therefore \left(1 - \frac{u^2}{c^2}\right)^{-1/2} = 1 + \frac{1}{2}\frac{u^2}{c^2}$$

Hence the equation (3.12) becomes,

$$E_k = m_0c^2 \left[1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right]$$

$$= m_0c^2 \left[\frac{1}{2}\frac{u^2}{c^2}\right]$$

$$= \frac{1}{2} m_0u^2$$

This is nothing but Newtonian expression for kinetic energy.

3.6 RELATIVISTIC RELATION BETWEEN ENERGY AND MOMENTUM, ($E^2 = p^2c^2 + m_0^2 c^4$) :

In Relativistic Mechanics,

$$E = mc^2$$

$$= \frac{m_0c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \quad \text{...(3.13)}$$

$$p = mu$$

$$= \frac{m_0u}{\sqrt{1 - \frac{u^2}{c^2}}} \quad \text{...(3.14)}$$

In order to obtain relation between E and p we shall eliminate the common variable quantity u as follows :

Squaring both sides of equation (3.14),

$$p^2 = \frac{m_0^2 u^2}{1 - \frac{u^2}{c^2}}$$

$$\therefore p^2 - \frac{p^2 u^2}{c^2} = m_0^2 u^2$$

$$\therefore p^2c^2 - p^2u^2 = m_0^2 u^2 c^2$$

$$\therefore u^2 (m_0^2 c^2 + p^2) = p^2 c^2$$

$$\therefore u^2 = \frac{p^2 c^2}{m_0^2 c^2 + p^2} \qquad ...(3.15)$$

Squaring both sides of equation (3.13)

$$E^2 = \frac{m_0^2 c^4}{1 - \frac{u^2}{c^2}}$$

Substituting the value of u^2 from the equation (3.15),

$$E^2 = \frac{m_0^2 c^4}{1 - \frac{1}{c^2} \cdot \frac{p^2 c^2}{m_0^2 c^2 + p^2}}$$

$$= \frac{m_0^2 c^4 (m_0^2 c^2 + p^2)}{m_0^2 c^2 + p^2 - p^2}$$

$$= c^2 (m_0^2 c^2 + p^2)$$

$$= p^2 c^2 + m_0^2 c^4$$

Thus the Relativistic relation between energy and momentum is

$$\boxed{E^2 = p^2 c^2 + m_0^2 c^4}$$

A Mnemonic Device : The above equation can be remembered using the following mnemonic device (Fig. 3.3) :

Let us represent geometrically the total energy E by the hypotenuse AC, the rest energy m_0c^2 by the side AB and pc by the side BC of the right angled ΔABC.

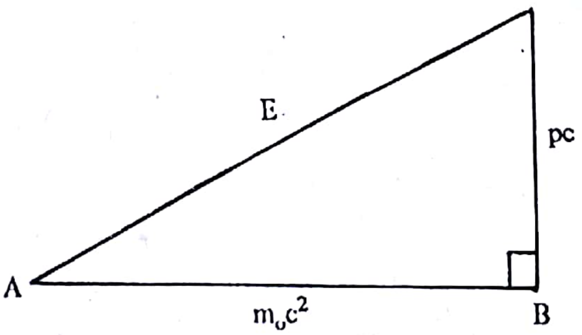


Fig. 3.3 A mnemonic device to remember Relativistic relation between energy and momentum

Then by Pythagoras' Theorem,

$$AC^2 = AB^2 + BC^2.$$

$$\therefore E^2 = m_0^2 c^4 + p^2c^2.$$

i.e. $E^2 = p^2c^2 + m_0^2 c^4.$

3.7 TO DERIVE THE EQUATION $\frac{dE}{dp} = u$:

The Relativistic relation between the energy and momentum is :

$$E^2 = p^2c^2 + m_0^2c^4$$

Differentiating both sides with respect to p,

$$\frac{d}{dp}(E^2) = \frac{d}{dp}(p^2c^2) + \frac{d}{dp}(m_0^2c^4)$$

$$\therefore \left[\frac{d}{dE}(E^2)\right]\frac{dE}{dp} = c^2\frac{d}{dp}(p^2) + 0$$

$$\therefore 2E \cdot \frac{dE}{dp} = 2pc^2$$

$$\therefore \frac{dE}{dp} = \frac{pc^2}{E} = \frac{muc^2}{mc^2} = u$$

Thus, we have $\frac{dE}{dp} = u$.

Note : It is interesting to note that the same equation is valid for a free particle in Classical Mechanics.

For a free particle, the total energy is its kinetic energy.

i.e. $E = E_k = \frac{1}{2}mu^2 = \frac{m^2u^2}{2m} = \frac{p^2}{2m}$

Differentiating both sides with respect to p,

$$\frac{dE}{dp} = \frac{d}{dp}\left(\frac{p^2}{2m}\right) = \frac{2p}{2m} = \frac{p}{m} = \frac{mu}{m} = u.$$

i.e. $\frac{dE}{dp} = u.$

3.8 LONGITUDINAL MASS AND TRANSVERSE MASS :

A force $\vec{F}$ is acting on stationary particle having resting mass m_0. The particle attains a velocity $\vec{u}$. Let its corresponding mass be m. Then by Newton's Second law of motion,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$= \frac{d}{dt}\left(m\vec{u}\right)$$

$$= m\frac{d\vec{u}}{dt} + \vec{u}\cdot\frac{dm}{dt}$$

$$= m\vec{a} + \vec{u}\cdot\frac{dm}{dt} \qquad \text{...(3.16)}$$

Now, total energy E of the particle is given by

$$E = m_0c^2 + E_k$$

$$\therefore mc^2 = m_0c^2 + E_k$$

$$\therefore m = m_0 + \frac{E_k}{c^2}$$

Differentiating both sides with respect to t,

$$\frac{dm}{dt} = \frac{dm_0}{dt} + \frac{d}{dt}\left(\frac{E_k}{c^2}\right)$$

$$= 0 + \frac{1}{c^2}\frac{dE_k}{dt}$$

$$= \frac{1}{c^2}\frac{dE_k}{dt}$$

Now, dE_k = change in the kinetic energy of the particle.

= work done by the external force on the particle

$$= dW = \vec{F}\cdot d\vec{r}.$$

$$\therefore \frac{dm}{dt} = \frac{1}{c^2}\frac{\vec{F}\cdot d\vec{r}}{dt} = \frac{1}{c^2}\vec{F}\cdot\frac{d\vec{r}}{dt} = \frac{1}{c^2}\left(\vec{F}\cdot\vec{u}\right)$$

$\therefore$ Equation (3.16) becomes

$$\vec{F} = m\vec{a} + \vec{u}\,\frac{1}{c^2}\left(\vec{F}\cdot\vec{u}\right) = m\vec{a} + \frac{\left(\vec{F}\cdot\vec{u}\right)}{c^2}\,\vec{u}$$

$$\therefore m\vec{a} = \vec{F} - \frac{\left(\vec{F}\cdot\vec{u}\right)}{c^2}\vec{u}$$

$$\therefore \vec{a} = \frac{\vec{F}}{m} - \frac{\left(\vec{F}\cdot\vec{u}\right)}{mc^2}\vec{u} \qquad \text{...(3.17)}$$

This equation shows that in Relativistic Mechanics the acceleration $\vec{a}$ is not having the same direction as the force $\vec{F}$ because the second term is having the direction of the velocity $\vec{u}$ of the particle (In Classical Mechanics $\vec{F} = m\vec{a}$. Hence in every motion the acceleration $\vec{a}$ has the same direction as the force $\vec{F}$). However, there are two cases in which $\vec{a}$ would have the same direction as $\vec{F}$.

Case I : When the force $\vec{F}$ is perpendicular to the velocity $\vec{u}$,

$$\vec{F}\cdot\vec{u} = F\,u\cos 90° = 0$$

The equation (3.17) reduces to

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\text{i.e. } \vec{F} = m\vec{a} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\vec{a}\,.$$

This force $\vec{F}$ and hence the acceleration $\vec{a}$ are perpendicular to the velocity $\vec{u}$ i.e. perpendicular to the direction of motion. Hence the above equation is written as

$$\vec{F_\perp} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \vec{a_\perp}$$

In the sense of Classical Mechanics the quantity $\frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$ is the mass experienced when the force acts in the direction perpendicular to the direction of motion. It is called **Transverse mass.**

Case II. When the force $\vec{F}$ is parallel to the velocity $\vec{u}$,

$$\vec{F} \cdot \vec{u} = F\,u \cos 0° = F\,u$$

The equation (3.17) becomes,

$$\vec{a} = \frac{\vec{F}}{m} - \frac{(Fu)}{mc^2} \vec{u}$$

Taking magnitude of both the sides,

$$a = \frac{F}{m} - \frac{(Fu)}{mc^2} \cdot u = \frac{F}{m} \left[1 - \frac{u^2}{c^2}\right]$$

$$\therefore F = \frac{ma}{1 - \frac{u^2}{c^2}} = \frac{m_0}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} a$$

$$\therefore \vec{F} = \frac{m_0}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} \vec{a}$$

This force $\vec{F}$ and hence the acceleration $\vec{a}$ are parallel to the velocity $\vec{u}$ i.e. parallel to the direction of motion. Hence the above equation is written as

$$\vec{F_{\parallel}} = \frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\vec{a_{\parallel}}$$

In the sense of Classical Mechanics the quantity $\frac{m_0}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}$ is the mass experienced when the force acts in the direction parallel to the direction of motion. It is called **Longitudinal mass.**

3.9 LORENTZ TRANSFORMATIONS OF MOMENTUM AND ENERGY

We have defined the relativistic momentum as

$$\vec{p} = m\vec{u} = \frac{m_0}{\sqrt{1-u^2/c^2}}\vec{u}$$

This equation refers to the frame S. In the frame S' the relativistic momentum is given by

$$\vec{p'} = m'\vec{u'} = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}\vec{u'}$$

Now, we know the transformation equations for the components of the velocity $\vec{u}$. Hence to get the transformation equations for components of momentum we shall first obtain the transformation equation for $\frac{1}{\sqrt{1-u^2/c^2}}$ as follows :

In the frame S', $u'^2 = u_x'^2 + u_y'^2 + u_z'^2$

$$\text{Now, } u'_x = \frac{u_x - V}{1-\frac{V.u_x}{c^2}}\ ;\ u'_y = \frac{u_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V.u_x}{c^2}}\ ;\ u'_z = \frac{u_z\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V.u_x}{c^2}}$$

$$\therefore u^2 = \frac{(u_x-V)^2}{\left(1-\frac{V.u_x}{c^2}\right)^2} + \frac{u_y^2\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V.u_x}{c^2}\right)^2} + \frac{u_z^2\left(1-\frac{V^2}{c^2}\right)}{\left(1-\frac{V.u_x}{c^2}\right)^2}$$

$$= \frac{u_x^2 - 2Vu_x + V^2 + u_y^2 - \frac{u_y^2V^2}{c^2} + u_z^2 - \frac{u_z^2V^2}{c^2}}{\left(1-\frac{V.u_x}{c^2}\right)^2}$$

$$= \frac{u_x^2 + u_y^2 + u_z^2 - 2Vu_x + V^2 - \frac{u_y^2 V^2}{c^2} - \frac{u_z^2 V^2}{c^2}}{\frac{(c^2 - V \cdot u_x)^2}{c^4}}$$

$$= \frac{u^2 - 2Vu_x + V^2 - \frac{u_y^2 V^2}{c^2} - \frac{u_z^2 V^2}{c^2}}{(c^2 - Vu_x)^2} \times c^4$$

$$\therefore \frac{u'^2}{c^2} = \frac{u^2c^2 - 2Vu_xc^2 + V^2c^2 - u_y^2 V^2 - u_z^2 V^2}{(c^2 - Vu_x)^2}$$

$$\therefore 1 - \frac{u'^2}{c^2} = 1 - \frac{u^2c^2 - 2Vu_xc^2 + V^2c^2 - u_y^2 \cdot V^2 - u_z^2 V^2}{(c^2 - Vu_x)^2}$$

$$= \frac{(c^2 - Vu_x)^2 - u^2c^2 + 2Vu_xc^2 - V^2c^2 + u_y^2 \cdot V^2 + u_z^2 V^2}{(c^2 - Vu_x)^2}$$

$$= \frac{c^4 - 2Vu_xc^2 + V^2 u_x^2 - u^2c^2 + 2Vu_xc^2 - V^2c^2 + u_y^2 \cdot V^2 + u_z^2 V^2}{(c^2 - Vu_x)^2}$$

$$= \frac{c^4 + V^2 u_x^2 - u^2c^2 - V^2c^2 + u_y^2 \cdot V^2 + u_z^2 V^2}{(c^2 - Vu_x)^2}$$

$$= \frac{c^4 + V^2 (u_x^2 + u_y^2 + u_z^2) - u^2c^2 - V^2c^2}{(c^2 - Vu_x)^2}$$

$$= \frac{c^4 + V^2u^2 - u^2c^2 - V^2c^2}{(c^2 - Vu_x)^2}$$

$$= \frac{c^2 (c^2 - u^2) - V^2 (c^2 - u^2)}{(c^2 - Vu_x)^2} = \frac{(c^2 - u^2)(c^2 - V^2)}{(c^2 - Vu_x)^2}$$

$$\therefore \frac{1}{1 - \frac{u'^2}{c^2}} = \frac{(c^2 - Vu_x)^2}{(c^2 - u^2)(c^2 - V^2)}$$

$$= \frac{c^4 \left(1 - \frac{Vu_x}{c^2}\right)^2}{c^2 \left(1 - \frac{u^2}{c^2}\right) c^2 \left(1 - \frac{V^2}{c^2}\right)} = \frac{\left(1 - \frac{Vu_x}{c^2}\right)^2}{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)}$$

$$\therefore \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{1 - \frac{Vu_x}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{V^2}{c^2}}} \qquad \text{...(3.18)}$$

i) Taking component of momentum along X'-axis in the frame S',

$$p'_x = m'u'_x$$

$$= \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}} u_x'$$

$$= \frac{m_0\left(1-\frac{V.u_x}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{V^2}{c^2}}} \frac{u_x - V}{\left(1-\frac{V.u_x}{c^2}\right)} \quad \text{by equation (3.18)}$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \cdot \frac{u_x - V}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{m(u_x - V)}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{mu_x - mV}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{p_x - mV}{\sqrt{1-\frac{V^2}{c^2}}}$$

We shall express mass m in terms of energy E as follows :

$$E = mc^2 \qquad \therefore m = \frac{E}{c^2}$$

$$\therefore p'_x = \frac{p_x - V\left(\frac{E}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

ii) Taking component of momentum along Y'-axis in the frame S',

$$p'_y = m'u'_y$$

$$= \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}} \cdot \frac{u_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V.u_x}{c^2}}$$

$$= \frac{m_0\left(1 - \frac{Vu_x}{c^2}\right)}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{V^2}{c^2}}} \cdot \frac{u_y\sqrt{1 - \frac{V^2}{c^2}}}{\left(1 - \frac{V \cdot u_x}{c^2}\right)}$$

$$= \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \cdot u_y$$

$$= mu_y$$

$$= p_y$$

iii) Using the same method as in (ii), we can prove,

$$p_z' = p_z$$

iv) Considering energy relative to the frame S',

$$E' = m'c^2$$

$$= \frac{m_0}{\sqrt{1 - \frac{u'^2}{c^2}}} \cdot c^2$$

$$= \frac{m_0\left(1 - \frac{Vu_x}{c^2}\right)}{\sqrt{1 - \frac{u^2}{c^2}}\sqrt{1 - \frac{V^2}{c^2}}} \cdot c^2$$

$$= \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \cdot \frac{(c^2 - Vu_x)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{m\,(c^2 - Vu_x)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{mc^2 - V\,(mu_x)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$= \frac{E - Vp_x}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Dividing both the sides by c^2 (to get the equation in standard form),

$$\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{V \cdot p_x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Thus the Lorentz transformation equations for momentum components and energy are

$$\text{i)}\quad p'_x = \frac{p_x - V\left(\frac{E}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \left[x' = \frac{x - Vt}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

$$\text{ii)}\quad p'_y = p_y \qquad [y' = y]$$

$$\text{iii)}\quad p'_z = p_z \qquad [z' = z]$$

$$\text{iv)}\quad \frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{Vp_x}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \left[t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}\right]$$

On comparing these transformation equations with the corresponding equations given in the bracket on right, we find that the momentum components transform in the same manner as the respective space co-ordinates and energy transforms in the same manner as the time co-ordinate. Hence these equations can be easily obtained from Lorentz transformation equations of the respective spac-time co-ordinates using the following correspondence:

$$x \rightarrow p_x, \qquad y \rightarrow p_y, \qquad z \rightarrow p_z, \qquad t \rightarrow \frac{E}{c^2}$$

We have observed in the second chapter (See §2.9) that in Special theory of Relativity, space and time co-ordinates are intimately and cohesively connected with each other. Moreover they can be converted into each other during co-ordinate transformations from one frame to another frame. The same is true for energy and momentum also. In Relativistic Mechanics, the energy and momentum are interdependent and can be converted into each other during co-ordinate transformations.

3.10 LORENTZ TRANSFORMATIONS OF MASS :

In the frame S' the mass of a particle is given by

$$m' = \frac{m_0}{\sqrt{1-\frac{u'^2}{c^2}}}$$

By equation (3.18),

$$\frac{1}{\sqrt{1-\frac{u'^2}{c^2}}} = \frac{1 - \frac{V.u_x}{c^2}}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore m' = \frac{m_0\left(1-\frac{Vu_x}{c^2}\right)}{\sqrt{1-\frac{u^2}{c^2}}\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}\cdot\frac{\left(1-\frac{Vu_x}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \frac{m\left(1-\frac{Vu_x}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

This is the transformation equation for mass.

Problem :

Show that the quantity

$$p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$$

remains invariant under Lorentz transformations of momentum and energy.

Solution :

Lorentz transformation equations for momentum and energy are

$$p_x' = \frac{p_x - \frac{VE}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}, \qquad p_y' = p_y, \qquad p_z' = p_z$$

$$\frac{E'}{c^2} = \frac{\frac{E}{c^2} - \frac{Vp_x}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \text{i.e. } E' = \frac{E - Vp_x}{\sqrt{1-\frac{V^2}{c^2}}}.$$

In the frame S', the quantity has the form,

$$p_x'^2 + p_y'^2 + p_z'^2 - \frac{E'^2}{c^2}$$

$$= \frac{(p_x - VE/c^2)^2}{\left(1-\frac{V^2}{c^2}\right)} + p_y^2 + p_z^2 - \frac{(E - Vp_x)^2}{c^2\left(1-\frac{V^2}{c^2}\right)}$$

$$= \frac{c^2\left(p_x - \frac{VE}{c^2}\right)^2 - (E - Vp_x)^2}{c^2\left(1 - \frac{V^2}{c^2}\right)} + p_y^2 + p_z^2$$

$$= \frac{c^2\left(p_x^2 - \frac{2EVp_x}{c^2} + \frac{V^2E^2}{c^4}\right) - \left(E^2 - 2EVp_x + V^2p_x^2\right)}{c^2 - V^2} + p_y^2 + p_z^2$$

$$= \frac{c^2p_x^2 - 2EVp_x + V^2E^2/c^2 - E^2 + 2EVp_x - V^2p_x^2}{c^2 - V^2} + p_y^2 + p_z^2$$

$$= \frac{c^2p_x^2 + \frac{V^2E^2}{c^2} - E^2 - V^2p_x^2}{c^2 - V^2} + p_y^2 + p_z^2$$

$$= \frac{p_x^2(c^2 - V^2) + E^2\left(\frac{V^2}{c^2} - 1\right)}{c^2 - V^2} + p_y^2 + p_z^2$$

$$= \frac{p_x^2(c^2 - V^2)}{c^2 - V^2} + \frac{E^2\left(\frac{V^2 - c^2}{c^2}\right)}{c^2 - V^2} + p_y^2 + p_z^2$$

$$= p_x^2 - \frac{E^2}{c^2} + p_y^2 + p_z^2$$

$$= p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$$

Thus the value of the quantity $p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$ remains the same in both the frames S and S'. Hence the quantity remains invariant under Lorentz transformations of momentum and energy.

Remark : This invariance is analogous to the invariance of $x^2 + y^2 + z^2 - c^2t^2$ under Lorentz transformations of space and time co-ordinates(as we have seen in the second chapter).

3.11 LORENTZ TRANSFORMATIONS OF FORCE :

By Newton's Second law of motion,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The components of force relative to the frame S are,

$$F_x = \frac{dp_x}{dt}, \qquad F_y = \frac{dp_y}{dt} \qquad F_z = \frac{dp_z}{dt}$$

The components of force relative to the frame S' are,

$$F'_x = \frac{dp'_x}{dt'}, \qquad F'_y = \frac{dp'_y}{dt'} \qquad F'_z = \frac{dp'_z}{dt'}$$

i) $F'_x = \dfrac{dp'_x}{dt'}$

The Lorentz transformation equation for X-component of momentum is

$$p'_x = \frac{p_x - V\left(\frac{E}{c^2}\right)}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore dp'_x = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\, d\left[p_x - \frac{VE}{c^2}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[dp_x - \frac{V}{c^2}\, dE\right]$$

Now, dE = change in the energy of the particle.
= work done by the external force on the particle.

$$= \vec{F}\cdot d\vec{r}.$$

$$\therefore dp'_x = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[dp_x - \frac{V}{c^2}\left(\vec{F}\cdot d\vec{r}\right)\right] \qquad \ldots(3.18)$$

The Lorentz transformation equation for the time co-ordinate is

$$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$\therefore dt' = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\, d\left[t - \frac{Vx}{c^2}\right]$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[dt - \frac{V}{c^2}dx\right]$$

$$= \frac{dt}{\sqrt{1-\frac{V^2}{c^2}}}\left[1 - \frac{V}{c^2}\frac{dx}{dt}\right]$$

$$= \frac{dt}{\sqrt{1-\frac{V^2}{c^2}}}\left[1 - \frac{Vu_x}{c^2}\right] \qquad \text{...(3.19)}$$

Using equation (3.18) and (3.19) we get the expression for F_x' as follows :

$$F_x' = \frac{dp_x'}{dt'}$$

$$= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left[dp_x - \frac{V}{c^2}\left(\vec{F}\cdot\vec{dr}\right)\right] \cdot \frac{\sqrt{1-\frac{V^2}{c^2}}}{dt\left(1-\frac{Vu_x}{c^2}\right)}$$

$$= \frac{1}{\left(1-\frac{Vu_x}{c^2}\right)}\left[\frac{dp_x}{dt} - \frac{V}{c^2}\left(\vec{F}\cdot\frac{\vec{dr}}{dt}\right)\right]$$

$$= \frac{F_x - \frac{V}{c^2}\left(\vec{F}\cdot\vec{u}\right)}{1-\frac{Vu_x}{c^2}}$$

ii) $F_y' = \dfrac{dp_y'}{dt'}$

The Lorentz transformation equation for Y-component of momentum is

$$p_y' = p_y$$

$$\therefore dp_y' = dp_y \qquad \text{...(3.20)}$$

Using equation (3.19) and (3.20) we get the expression for F'_y as follows :

$$F'_y = \frac{dp'_y}{dt'}$$

$$= \frac{dp_y}{dt\left(1 - \frac{Vu_x}{c^2}\right)} \cdot \sqrt{1 - \frac{V^2}{c^2}}$$

$$= \frac{\frac{dp_y}{dt}\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}} = \frac{F_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$$

iii) Using the same method as in (ii) we can prove,

$$F'_z = \frac{F_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$$

Thus the Lorentz transformation equations for the components of force are,

$$\boxed{\begin{aligned} F'_x &= \frac{F_x - \frac{V}{c^2}\left(\vec{F} \cdot \vec{u}\right)}{1 - \frac{Vu_x}{c^2}} \\ F'_y &= \frac{F_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}} \\ F'_z &= \frac{F_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}} \end{aligned}}$$

Remarks : In Newtonian Relativity,(as we have proved in section 1.12 of the first chapter), $F' = F$. i.e. the force is invariant under Galilean transformations. This is not so in Relativistic Mechanics. The force is a relative quantity and varies from one inertial frame of reference to another inertial frame.

However, you can see easily, for small values of u and V, the above equations reduce to $F'_x = F_x$, $F'_y = F_y$ and $F'_z = F_z$ i.e. $F' = F$, as we naturally expect.

3.12 A LIST OF IMPORTANT FORMULAE OF THE CHAPTER :

i) $m = \dfrac{m_0}{\sqrt{1-\dfrac{u^2}{c^2}}}$

ii) $\vec{p} = m\vec{u}$

iii) $\vec{F} = \dfrac{d\vec{p}}{dt}$

iv) $E = mc^2$

v) $E = m_0c^2 + E_k$

vi) For photons, $m_0 = 0$

vii) $E_k = m_0c^2\left[\dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}} - 1\right]$

viii) $E^2 = p^2c^2 + m_0^2c^4$

ix) Correspondence between momentum - energy and space-time co-ordinates :

$x \rightarrow p_x, \quad y \rightarrow p_y, \quad z \rightarrow p_z, \quad t \rightarrow \frac{E}{c^2}.$

x) $F'_x = \dfrac{F_x - \dfrac{V}{c^2}\left(\vec{F}\cdot\vec{u}\right)}{1-\dfrac{Vu_x}{c^2}}$

$F'_y = \dfrac{F_y\sqrt{1-\dfrac{V^2}{c^2}}}{1-\dfrac{Vu_x}{c^2}}$

$F'_z = \dfrac{F_z\sqrt{1-\dfrac{V^2}{c^2}}}{1-\dfrac{Vu_x}{c^2}}$

Problem : What should be the velocity of a particle so that its mass is ten times its rest mass?

Given : $u = ?$ $m = 10\, m_0$.

Solution :

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\therefore 10m_0 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\therefore \sqrt{1-\frac{u^2}{c^2}} = \frac{1}{10} = 0.1$$

$$\therefore 1-\frac{u^2}{c^2} = 0.01$$

$$\therefore \frac{u^2}{c^2} = 1-0.01 = 0.99$$

$$\therefore \frac{u}{c} = \sqrt{0.99} = 0.9949$$

$$\therefore u = 0.9949c$$

Problem : What is the ratio of the mass of a particle to its rest mass if it is moving with velocity 0.8c.

Given : $\frac{m}{m_0} = ?$, $u = 0.8c$.

Solution :

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\therefore \frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$= \frac{1}{\sqrt{1-0.64}} = \frac{1}{\sqrt{0.36}} = \frac{1}{0.6} = 1.667$$

$$\text{i.e. } \frac{m}{m_0} = 1.667$$

Problem : The density of Gold is 19.3 gm/cm^3 when it is at rest relative to an observer. What is its density when it is moving with the relative velocity 0.9c?

Given : $\rho_0 = 19.3$ gm/cm^3, $\rho = ?$, $u = 0.9c$.

Solution :

Density of a substance is given by

$$\rho = \frac{m}{V} = \frac{m}{\Delta x \cdot \Delta y \cdot \Delta z}$$

Now, $m = \dfrac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$

By Lorentz - Fritzgerald length contraction,

$$\Delta x = (\Delta x)_0 \sqrt{1 - \frac{u^2}{c^2}} \; ; \; \Delta y = (\Delta y)_0 \; ; \; \Delta z = (\Delta z)_0$$

$$\therefore \rho = \frac{\dfrac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}}{(\Delta x)_0 \sqrt{1 - \frac{u^2}{c^2}} \cdot (\Delta y)_0 \cdot (\Delta z)_0}$$

$$= \frac{1}{1 - \frac{u^2}{c^2}} \cdot \frac{m_0}{(\Delta x)_0 \cdot (\Delta y)_0 \cdot (\Delta z)_0}$$

$$= \frac{\rho_0}{1 - \frac{u^2}{c^2}}$$

$$= \frac{19.3}{1 - 0.81} = \frac{19.3}{0.19} = 101.6 \text{ gm/cm}^3$$

Problem : The momentum of an electron observed in a magnetic deflection experiment is $3m_0c$, where m_0 is rest mass of the electron and c is the velocity of light. Find the velocity of the electron and its relativistic kinetic energy.

Given : $p = 3m_0c$, $u = ?$, $E_k = ?$

Solution :

$$p = 3m_0c$$

$$\text{Now,} \quad p = mu = \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$\therefore \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}} = 3m_0c$$

$$\therefore \frac{u^2}{1 - \frac{u^2}{c^2}} = 9c^2$$

$$\therefore u^2 = 9c^2 - 9u^2$$

$$\therefore 10u^2 = 9c^2 \quad \therefore u^2 = 0.9\,c^2$$

$$\therefore u = \sqrt{0.9}\,c = 0.9487c$$

$$E_k = m_0c^2\left[\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right]$$

$$= m_0c^2\left[\frac{1}{\sqrt{1-0.9}} - 1\right]$$

$$= m_0c^2\left[\sqrt{10} - 1\right]$$

$$= 2.162\ m_0c^2$$

Problem : How much energy is released in the explosion of a fission bomb containing 5 kg of a fissionable material? Assume that 0.1 percent of rest mass is converted into energy.

Solution : If total mass is 100 kg then 0.1 kg is converted.

$\therefore$ If total mass is 5 kg then $\frac{0.1 \times 5}{100} = 0.005$ kg is converted.

$\therefore m = 0.005$ kg.

The equivalent amount of energy is given by

$$E = mc^2$$

$$= 0.005 \times (3 \times 10^8)^2 = 0.045 \times 10^{16} = 4.5 \times 10^{14}\ \text{J}.$$

$\therefore$ Energy released in the explosion $= 4.5 \times 10^{14}$ J.

Problem : Find the energy equivalent to 1 a.m.u. in MeV.

Given : 1 a.m.u. $= 1.66 \times 10^{-27}$kg

$c = 2.998 \times 10^8$ m/s

1 eV $= 1.602 \times 10^{-19}$ J

Solution :

$$E = mc^2$$

$$= 1.66 \times 10^{-27} \times (2.998 \times 10^8)^2$$

$$= 1.66 \times (2.998)^2 \times 10^{-11}\ \text{J}$$

$$= \frac{1.66 \times (2.998)^2 \times 10^{-11}}{1.602 \times 10^{-19}}\ \text{eV}$$

$$= \frac{1.66 \times (2.998)^2 \times 10^{8}}{1.602}\ \text{eV}$$

$$= 9.313 \times 10^8\ \text{eV}$$

$$= 931.3 \times 10^6\ \text{eV}$$

$$= 931.3\ \text{MeV}$$

Problem : What is the mass of a cosmic ray particle having energy 15 MeV. Take $c = 3 \times 10^8$ m/s.

Given : $m = ?$

$$E = 15 \text{ MeV} = 15 \times 10^6 \text{eV} = 15 \times 10^6 \times 1.6 \times 10^{-19} \text{J}$$
$$= 15 \times 1.6 \times 10^{-13} \text{ J}$$

Solution :

$$E = mc^2$$

$$\therefore m = \frac{E}{c^2} = \frac{15 \times 1.6 \times 10^{-13}}{(3 \times 10^8)^2}$$

$$= \frac{15 \times 1.6}{9} \times 10^{-29} = 2.667 \times 10^{-29} \text{ kg.}$$

Problem : Calculate the velocity of a body when its total energy is two times its rest energy.

Given : $E = 2m_0c^2$, $u = ?$

Solution :

$$E = mc^2$$

$$\therefore 2m_0c^2 = mc^2$$

$$\therefore m = 2m_0$$

$$\therefore \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} = 2m_0$$

$$\therefore \sqrt{1 - \frac{u^2}{c^2}} = \frac{1}{2}$$

$$\therefore 1 - \frac{u^2}{c^2} = \frac{1}{4}$$

$$\therefore \frac{u^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore u = \sqrt{\frac{3}{4}}\, c = 0.8660c$$

Problem : The nucleus of helium atom consists of 2 protons and 2 neutrons bound together by nuclear forces. The rest masses of the following particles are as under :

Mass of helium nucleus = 4.003871 a.m.u.
Mass of proton = 1.007825 a.m.u.
Mass of neutron = 1.008665 a.m.u.

Find the binding energy of one nucleus of helium in MeV.

Take 1 a.m.u. = 1.66×10^{-27} kg.

Velocity of light = 3×10^8 m/s.

Solution :

Mass of two protons = 2×1.007825 = 2.015650 a.m.u.

Mass of two neutrons = 2×1.008665 = 2.017330 a.m.u.

∴ Total mass of two protons & two neutrons = 4.032980 a.m.u.

Mass of one nucleus of helium = 4.003871 a.m.u.

∴ Mass defect for one nucleus of helium

$$= m = 4.032980 - 4.003871$$

$$= 0.029109 \text{ a.m.u.}$$

$$= 0.029109 \times 1.66 \times 10^{-27} \text{ Kg.}$$

∴ Binding energy of one nucleus of helium

$$= E = mc^2$$

$$= 0.029109 \times 1.66 \times 10^{-27} \times (3 \times 10^8)^2$$

$$= 0.029109 \times 1.66 \times 9 \times 10^{-11} \text{ J}$$

$$= \frac{0.029109 \times 1.66 \times 9 \times 10^{-11}}{1.6 \times 10^{-19}} \text{ eV}$$

$$= 27.18 \times 10^6 \text{ eV}$$

$$= 27.18 \text{ MeV}$$

Problem : A nucleus of mass 235.124 a.m.u. breaks into two nuclei of masses 97.936 a.m.u. and 135.951 a.m.u. and a neutron of mass 1.009 a.m.u. Calculate the energy released when 1 milligram of such nuclei disintegrate. Given : 1 a.m.u. = 1.66×10^{-27} kg.

Solution :

Total initial mass of one nucleus = 235.124 a.m.u.

Total final mass = 97.936 + 135.951 + 1.009

= 234.896 a.m.u.

∴ Loss in mass for one nucleus = 235.124 - 234.896

= 0.228 a.m.u.

= $0.228 \times 1.66 \times 10^{-27}$ kg.

Now 1 milligram = 10^{-6} kg.

Mass of one original nucleus = $235.124 \times 1.66 \times 10^{-27}$ kg.

∴ Total number of such nuclei in 1 mg.

$$= \frac{10^{-6}}{235.124 \times 1.66 \times 10^{-27}}$$

∴ Total loss in mass for 1 mg.

$$= \frac{10^{-6}}{235.124 \times 1.66 \times 10^{-27}} \times 0.228 \times 1.66 \times 10^{-27}$$

$$= \frac{0.228 \times 10^{-6}}{235.124} = 9.697 \times 10^{-10} \text{ kg.}$$

∴ Total energy released $= E = mc^2$

$$= 9.697 \times 10^{-10} \times (3 \times 10^8)^2$$

$$= 8.727 \times 10^7 \text{ J.}$$

Problem : The earth receives energy from the sun at the rate of 1.34×10^3 Watt/m^2. The distance between the earth and the sun is 1.5×10^8 km. Find the annual loss of the mass of the sun.

Solution :

The energy emitted by the sun is distributed over a sphere of radius $r = 1.5 \times 10^8$ km $= 1.5 \times 10^{11}$ m. The area of the sphere is given by

$$A = 4\pi r^2$$

$$= 4 \times 3.142 \times (1.5 \times 10^{11})^2 = 28.28 \times 10^{22} \text{ m}^2$$

The energy received at the earth $= 1.34 \times 10^3$ Watt/m^2

$= 1.34 \times 10^3$ J/sec/m^2

∴ The energy received per square metre per sec. of the above sphere $= 1.34 \times 10^3$ J

∴ Total energy received over the whole sphere in 1 year

$$= 1.34 \times 10^3 \times 28.28 \times 10^{22} \times 365 \times 24 \times 3{,}600$$

$$= 11.95 \times 10^{33} \text{ J.}$$

∴ Total energy emitted by the sun in 1 year

$$= E = 11.95 \times 10^{33} \text{J}$$

The corresponding mass (m) lost by the sun in 1 year is given by

$$m = \frac{E}{c^2} = \frac{11.95 \times 10^{33}}{(3 \times 10^8)^2} = 1.328 \times 10^{17} \text{ kg.}$$

Problem : Find the mass and kinetic energy of a photon of wavelength 5000 A.U. Take Planck's constant = $h = 6.63 \times 10^{-34}$ J.s., velocity of light = $c = 3 \times 10^{8}$ m/s.

Given : $m = ?$, $E_k = ?$, $\lambda = 5000$ A.U. $= 5000 \times 10^{-10}$m.

Solution :

The energy of the photon is given by

$$\epsilon = h\nu = \frac{hc}{\lambda}$$

If the mass equivalent to this energy is m them

$$\epsilon = mc^2$$

$$\therefore m = \frac{\epsilon}{c^2} = \frac{hc}{\lambda} \times \frac{1}{c^2} = \frac{h}{\lambda c}$$

$$\therefore m = \frac{6.63 \times 10^{-34}}{5{,}000 \times 10^{-10} \times 3 \times 10^{8}}$$

$$= \frac{6.63 \times 10^{-34}}{150} = 4.42 \times 10^{-36} \text{ kg.}$$

Moreover,

$$E = m_0c^2 + E_k$$

The rest mass (m_0) of photon is zero.

$$\therefore E_k = E = h\nu = \frac{hc}{\lambda}$$

$$\therefore E_k = \frac{6.63 \times 10^{-34} \times 3 \times 10^{8}}{5{,}000 \times 10^{-10}}$$

$$= \frac{6.63 \times 3}{5{,}000} \times 10^{-16} = 3.979 \times 10^{-19} \text{ J}$$

Problem : An electron has kinetic energy equal to twice its rest energy. Find its speed in terms of c.

Given : $E_k = 2m_0c^2$, $u = ?$

Solution :

$$\therefore E_k = m_0c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right]$$

$$\therefore 2m_0c^2 = m_0c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right]$$

$$\therefore \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1 = 2$$

$$\therefore \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 3$$

$$\therefore 1-\frac{u^2}{c^2} = \frac{1}{9}$$

$$\therefore \frac{u^2}{c^2} = 1-\frac{1}{9} = \frac{8}{9}$$

$$\therefore \frac{u}{c} = \sqrt{\frac{8}{9}} = 0.9428$$

$$\therefore u = 0.9428c$$

Problem : An electron is accelerated in a synchrotron to a kinetic energy of 1 MeV. What is the mass of the electron in terms of its rest mass? What is the speed of the electron in terms of the speed of light? Rest mass of electron = 9.1×10^{-31} kg.

Given : $E_k = 1 \text{ MeV} = 10^6 \text{ eV} = 10^6 \times 1.6 \times 10^{-19} \text{ J} = 1.6 \times 10^{-13} \text{ J}.$

$m = ?,\ u = ?$

Solution :

$$E = m_0c^2 + E_k$$

$$\therefore mc^2 = m_0c^2 + E_k$$

$$\therefore m = m_0 + \frac{E_k}{c^2}$$

$$= m_0\left[1 + \frac{E_k}{m_0c^2}\right]$$

$$= m_0\left[1 + \frac{1.6\times10^{-13}}{9.1\times10^{-31}\times(3\times10^8)^2}\right]$$

$$= m_0\left[1 + \frac{1.6\times10^2}{9.1\times9}\right]$$

$$= m_0\,[1 + 1.954]$$

$$= 2.954\,m_0$$

Now,

$$m = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$\therefore \sqrt{1-\frac{u^2}{c^2}} = \frac{m_0}{m} = \frac{m_0}{2.954\,m_0} = \frac{1}{2.954}$$

$$\therefore 1 - \frac{u^2}{c^2} = \frac{1}{(2.954)^2} = 0.1146$$

$$\therefore \frac{u^2}{c^2} = 1 - 0.1146 = 0.8854$$

$$\therefore \frac{u}{c} = \sqrt{0.8854} = 0.9410$$

$$\therefore u = 0.9410c$$

Problem : An electron starting from rest gets accelerated when it moves through a potential difference of 10^5V. Calculate its mass and the speed at the end of the acceleration.

Given : $m_0 = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C.

Solution :

By definition of potential difference,

$$V = \frac{W}{q} \qquad \therefore W = V.q$$

$\therefore$ Work done by the electric field on the electron

$$= W = V.e$$
$$= 10^5 \times 1.6 \times 10^{-19} = 1.6 \times 10^{-14} \text{ J}$$

This work appears as kinetic energy of the electron.

$$\therefore E_k = 1.6 \times 10^{-14} \text{ J}$$

Now, $E = m_0c^2 + E_k$

$$\therefore mc^2 = m_0c^2 + E_k$$

$$\therefore m = m_0 + \frac{E_k}{c^2}$$

$$= 9.1 \times 10^{-31} + \frac{1.6 \times 10^{-14}}{(3 \times 10^8)^2}$$

$$= 9.1 \times 10^{-31} + 1.778 \times 10^{-31}$$

$$= 10.878 \times 10^{-31} \text{ kg.}$$

Now, $m = \dfrac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}}$

$$\therefore \sqrt{1 - \frac{u^2}{c^2}} = \frac{m_0}{m}$$

$$\therefore 1 - \frac{u^2}{c^2} = \left(\frac{m_0}{m}\right)^2 = \left(\frac{9.1 \times 10^{-31}}{10.878 \times 10^{-31}}\right)^2 = 0.6998$$

$$\therefore \frac{u^2}{c^2} = 1 - 0.6998 = 0.3002$$

$$\therefore \frac{u}{c} = \sqrt{0.3002} = 0.5479$$

$$\therefore u = 0.5479\,c = 0.5479 \times 3 \times 10^8$$

$$= 1.6437 \times 10^8 \text{ m/s}$$

Problem : Through what potential difference should an electron starting from rest be accelerated to attain the velocity of light c, according to Classical Mechanics? If an electron is accelerated through this potential difference what would be the velocity attained by the electron according to Relativistic Mechanics? Take $m_0 = 9.1 \times 10^{-31}$ kg, $e = 1.6 \times 10^{-19}$ C, $c = 3 \times 10^8$ m/s.

Solution :

Let the required potential difference be V.

By definition of potential difference,

$$V = \frac{W}{q} \qquad \therefore W = V.q$$

$\therefore$ Work done by the electric field on the electron $= W = V.e$

This work appears as kinetic energy of the electron.

$$\therefore E_k = Ve$$

Now, when the electron attains the velocity of light, according to Classical Mechanics, its kinetic energy is given by

$$E_k = \frac{1}{2} m_0 c^2$$

$$\therefore Ve = \frac{1}{2} m_0 c^2$$

$$\therefore V = \frac{m_0 c^2}{2e}$$

$$= \frac{9.1 \times 10^{-31} \times 9 \times 10^{16}}{2 \times 1.6 \times 10^{-19}} = \frac{9.1 \times 9}{3.2} \times 10^4 = 2.559 \times 10^5 \text{V}$$

Potential difference to attain the velocity of light = 2.559×10^5 V

According to Relativistic Mechanics,

$$E_k = m_0 c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right]$$

$$\therefore Ve = m_0 c^2 \left[\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right]$$

$$\therefore \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1 + \frac{Ve}{m_0 c^2}$$

$$= 1 + \frac{2.559 \times 10^5 \times 1.6 \times 10^{-19}}{9.1 \times 10^{-31} \times 9 \times 10^{16}}$$

$$= 1 + \frac{2.559 \times 1.6 \times 10}{81.9}$$

$$= 1 + 0.5$$

$$= 1.5$$

$$\therefore 1 - \frac{u^2}{c^2} = \frac{1}{(1.5)^2} = 0.4444$$

$$\therefore \frac{u^2}{c^2} = 1 - 0.4444 = 0.5556$$

$$\therefore \frac{u}{c} = 0.7454$$

$$\therefore u = 0.7454 \times 3 \times 10^8 = 2.2362 \times 10^8 \text{ m/s}$$

Velocity attained by the electron accelerated through the above potential difference = 2.2362×10^8 m/s. This is much less than $c = 3 \times 10^8$ m/s.

QUESTIONS

4

Relativistic Electromagnetism

In this chapter we shall develop Relativistic Electromagnetism. We have seen in the first chapter that the laws of electrodynamics are not invariant in all inertial frames of reference under Galilean transformations.

In the second chapter we have observed that the laws of electrodynamics are invariant in all inertial frames of reference under Lorentz transformations. i.e. the equations representing these laws have the same form in all inertial frames of reference as required by the first postulate of the Special theory of Relativity. In fact, this was the reason which led Einstein in 1905 to believe that the Lorentz transformation equations are the correct transformation equations for inertial frames of reference and they should replace the Galilean transformation equations.

Thus as regards electromagnetism, the basic equations representing different laws have covariant form and they do not require any modification. However, the quantities of electromagnetism vary from one inertial frame to another inertial frame but the laws governing them remain invariant. In this chapter we shall study the transformation properties of these quantities from one inertial frame to another inertial frame.

4.1 ACTION OF ELECTRIC AND MAGNETIC FIELDS ON AN ELECTRIC CHARGE :

Electric force on an electric charge :

A particle having charge q is lying in an electric field of intensity $\vec{E}$. Then a force $\vec{F}$ acts on the particle and is given by

$$\boxed{\vec{F} = q\vec{E}}$$

The magnitude of the force is $F = qE$. Its direction is the same as that of $\vec{E}$ if the charge is positive. If the charge is negative then the direction of $\vec{F}$ is opposite to that of $\vec{E}$.

Magnetic force on an electric charge :

A particle having charge q is moving with a velocity $\vec{u}$ in a magnetic field of induction $\vec{B}$. Then a force $\vec{F}$ acts on the particle and it is given by

$$\boxed{\vec{F} = q\vec{u} \times \vec{B}}$$

The magnitude of the force is given by

$$F = quB \sin\theta$$

where θ is the angle between $\vec{u}$ and $\vec{B}$. The force $\vec{F}$ is directed along the normal to the plane of $\vec{u}$ and $\vec{B}$. i.e. it is normal to $\vec{u}$ and $\vec{B}$. In case, if the particle is stationary relative to the magnetic field then $u = 0$ and no force is exerted by the magnetic field.

Electromagnetic force on an electric charge :

A particle having charge q is moving with a velocity $\vec{u}$ in space where electric field of intensity $\vec{E}$ and a magnetic field of induction $\vec{B}$ are present. Then a force $\vec{F}$ acts on the particle and is given by

$$\boxed{\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}}$$

This equation represents **Lorentz force law.**

4.2 INTERDEPENDENCE OF ELECTRIC AND MAGNETIC FIELDS :

Let us consider a charge q_s which may be called a source charge. It is moving with a velocity $\vec{u}$ along X-axis relative to the frame S. A moving charge is equivalent to a current and hence it gives rise to a magnetic field of induction $\vec{B}$ around itself, in addition to electric field.

This is observed by an observer from another frame of reference S'. Let the velocity of the frame S' be $V = u$ along X-axis relative to the frame S. Then the source charge appears to be stationary in the frame S'. Hence it does not give rise to the magnetic field in the frame S'. The observer in the frame S' finds that a charge q experiences force due to only the electric field of the source charge q_s. Thus by introducing suitable frame of reference we can eliminate the magnetic field. What appears as a mixture of electric and magnetic fields in one frame of reference is nothing but an electric field when observed from another frame.

Now, let the source charge q_s be stationary in the frame S. Hence it produces only electric field in that frame.

For the observer in the frame S', the charge is moving along X'-axis with velocity $-V$ and hence it produces a magnetic field also. i.e. what appears as purely an electric field in one frame is nothing but a mixture of electric and magnetic fields when observed from another frame.

VIMP [Thus in Relativity, the difference between an electric field and a magnetic field is relative and depends upon the frame of reference with respect to which the observations are made.]

4.3 LORENTZ TRANSFORMATIONS OF ELECTRIC INTENSITY $\vec{E}$ AND MAGNETIC INDUCTION $\vec{B}$:

A particle having a charge q is lying at a point in space where the intensity of the electric field is $\vec{E}$ and that of magnetic field is $\vec{B}$. The force acting on the particle relative to the frame S is given by

$$\vec{F} = q\vec{E} + q\vec{u} \times \vec{B} \qquad \ldots(4.1)$$

where $\vec{u}$ is the instantaneous velocity of the particle at that point.

According to the first postulate of the Special theory of Relativity, this equation representing the Lorentz force law must be invariant in all inertial frames of reference. Hence relative to the frame S' the force acting on the particle is given by

$$\vec{F'} = q\vec{E'} + q\,\vec{u'} \times \vec{B'} \qquad \text{...(4.2)}$$

[Note : In Relativistic Electromagnetism the charge q is considered to be an invariant quantity, because otherwise, the motion of the electrons in an atom would lead to a change in their electric charge. Due to this, the balance of positive and negative charges of an atom would be disturbed and the atom would no longer be neutral.]

For electric intensity $\vec{E}$:

In order to eliminate the magnetic field in the frame S' we shall select the velocity V of the frame S' so that the particle is stationary relative to the frame S'.

i.e. $\vec{u'} = \vec{0}$.

∴ Equation (4.2) becomes,

$$\vec{F'} = q\vec{E'}.$$

This equation can be written in component form as follows :

$$F'_x = qE'_x \qquad \text{...(4.3)}$$

$$F'_y = qE'_y \qquad \text{...(4.4)}$$

$$F'_z = qE'_z \qquad \text{...(4.5)}$$

Since the particle is stationary in the frame S', it must be moving with velocity V along X-axis relative to the frame S.

$\therefore u_x = V, \quad u_y = 0, \quad u_z = 0.$

$$\therefore \vec{u} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V & 0 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

∴ Equation (4.1) can be written in component form as follows :

$$F_x = qE_x + q\left(\vec{u} \times \vec{B}\right)_x$$

$$= qE_x + q\,(0)$$

$$= qE_x \qquad \text{...(4.3)'}$$

$$F_y = qE_y + q\left(\vec{u} \times \vec{B}\right)_y$$

$$= qE_y + q\,(0 - VB_z)$$

$$= qE_y - q\,V\,B_z \qquad \text{...(4.4)'}$$

$$F_z = qE_z + q\left(\vec{u} \times \vec{B}\right)_z$$

$$= qE_z + q\,(VB_y - 0)$$

$$= qE_z + qVB_y \qquad \text{...(4.5)'}$$

i) By Lorentz transformations of force, X-Component of force is given by

$$F'_x = \frac{F_x - \frac{V}{c^2}\left(\vec{F} \cdot \vec{u}\right)}{1 - \frac{V \, . \, u_x}{c^2}}$$

Taking inverse transformations,

$$F_x = \frac{F'_x + \frac{V}{c^2}\left(\vec{F'} \cdot \vec{u'}\right)}{1 + \frac{V \, . \, u'_x}{c^2}}$$

$$\text{Now, } \vec{u'} = \vec{0} \,.$$

$$\therefore u'_x = 0$$

$$\therefore F_x = \frac{F'_x + \frac{V}{c^2} \times 0}{1 + V \times \frac{0}{c^2}} = F'_x$$

By equations (4.3) and (4.3)' the above equation becomes,

$$qE_x = qE'_x \qquad \therefore E'_x = E_x$$

ii) By Lorentz transformations of force, Y-component of force is given by,

$$F_y' = \frac{F_y\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{Vu_x}{c^2}}$$

$$\therefore F_y = \frac{F_y'\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{Vu_x'}{c^2}} = F_y'\sqrt{1-\frac{V^2}{c^2}}$$

By equations (4.4) and (4.4)', the above equation becomes,

$$qE_y - qVB_z = qE_y'\sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore E_y' = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}}$$

We shall denote $\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$ by γ.

$$\therefore E_y' = \gamma\,[E_y - VBz]$$

iii) By Lorentz transformations of force, Z-component of force is given by

$$F_z' = \frac{F_z\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{Vu_x}{c^2}}$$

$$\therefore F_z = \frac{F_z'\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{Vu_x'}{c^2}} = F_z'\sqrt{1-\frac{V^2}{c^2}}$$

By equations (4.5) and (4.5)', the above equation becomes,

$$qE_z + qVB_y = qE_z'\sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore E_z' = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$= \gamma\,[E_z + VB_y]$$

Thus from (i) (ii) and (iii) we get

$$E'_x = E_x \qquad E'_y = \gamma[E_y - VB_z] \qquad E'_z = \gamma[E_z + VB_y]$$

The first equation shows that the intensity of the electric field remains the same in both the frames S and S' in the direction of relative motion between the two frames.

i.e $\vec{E'_{\parallel}} = \vec{E}_{\parallel}$

The second and the third equations show that there is a change in the intensity of the electric field in the direction perpendicular to the direction of relative motion.

Now, $$\vec{V} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V & 0 & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

Hence the last two equations can be written as

$$E'_y = \gamma\left[E_y + \left(\vec{V} \times \vec{B}\right)_y\right]$$

$$E'_z = \gamma\left[E_z + \left(\vec{V} \times \vec{B}\right)_z\right]$$

In general, $$\vec{E'_{\perp}} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right]$$

Hence the Lorentz transformations of electric intensity are

$$\boxed{\begin{aligned} \vec{E'_{\parallel}} &= \vec{E}_{\parallel} \\ \vec{E'_{\perp}} &= \gamma\left[\vec{E}_{\perp} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right] \end{aligned}}$$

For magnetic induction $\vec{B}$:

- In order that the magnetic induction $\vec{B'}$ survives in the frame S'we shall consider the particle to be moving with velocity u' along Y'-axis relative to the frame S'.

i.e. $\therefore u'_x = 0,\ u'_y = u',\ u'_z = 0.$

$$\text{Now, } \vec{u'} \times \vec{B'} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & u' & 0 \\ B'_x & B'_y & B'_z \end{vmatrix}$$

Hence equation (4.2) can be written in component form as follows :

$$\begin{aligned} F'_x &= qE'_x + q\left(\vec{u'} \times \vec{B'}\right)_x \\ &= qE'_x + q\left(u'B'_z - 0\right) \\ &= qE'_x + q\,.\,u'B'_z \qquad \text{...(4.6)} \end{aligned}$$

$$\begin{aligned} F'_y &= qE'_y + q\left(\vec{u'} \times \vec{B'}\right)_y \\ &= qE'_y + q\,(0 - 0) \\ &= qE'_y \qquad \text{...(4.7)} \end{aligned}$$

$$\begin{aligned} F'_z &= qE'_z + q\left(\vec{u'} \times \vec{B'}\right)_z \\ &= qE'_z + q\left(0 - u'B'_x\right) \\ &= qE'_z - q\,u'B'_x \qquad \text{...(4.8)} \end{aligned}$$

We shall find out now the components of the velocity of the particle relative to the frame S.

By Lorentz transformations of velocity,

$$u'_x = \frac{u_x - V}{1 - \frac{V.u_x}{c^2}} \qquad \therefore u_x = \frac{u'_x + V}{1 + \frac{V.u'_x}{c^2}} = \frac{0 + V}{1 + 0} = V$$

$$u'_y = \frac{u_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V.u_x}{c^2}} \qquad \therefore u_y = \frac{u'_y\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V.u'_x}{c^2}} = u'\sqrt{1 - \frac{V^2}{c^2}}$$

$$u'_z = \frac{u_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V.u_x}{c^2}} \qquad \therefore u_z = \frac{u'_z\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V.u'_x}{c^2}} = 0$$

$$\therefore \vec{u} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ V & u'\sqrt{1-\frac{V^2}{c^2}} & 0 \\ B_x & B_y & B_z \end{vmatrix}$$

$\therefore$ Equation (4.1) can be written in component form as follows :

$$F_x = qE_x + q\left(\vec{u} \times \vec{B}\right)_x .$$

$$= qE_x + q\left(B_z u'\sqrt{1-\frac{V^2}{c^2}} - 0\right)$$

$$= qE_x + qB_z u'\sqrt{1-\frac{V^2}{c^2}} \qquad \text{...(4.6)'}$$

$$F_y = qE_y + q\left(\vec{u} \times \vec{B}\right)_y$$

$$= qE_y + q\,(0 - VB_z)$$

$$= qE_y - qVB_z \qquad \text{...(4.7)'}$$

$$F_z = qE_z + q\left(\vec{u} \times \vec{B}\right)_z .$$

$$= qE_z + q\left(VB_y - B_x u'\sqrt{1-\frac{V^2}{c^2}}\right)$$

$$= qE_z + qVB_y - qB_x u'\sqrt{1-\frac{V^2}{c^2}} \qquad \text{...(4.8)'}$$

iv) By Lorentz transformations of force, X-component of force is given by

$$F_x' = \frac{F_x - \frac{V^2}{c^2}\left(\vec{F} \cdot \vec{u}\right)}{1 - \frac{V \cdot u_x}{c^2}}$$

Taking inverse transformations,

$$F_x = \frac{F_x' + \frac{V}{c^2}\left(\vec{F'} \cdot \vec{u'}\right)}{1 + \frac{Vu_x'}{c^2}} = \frac{F_x' + \frac{V}{c^2}\; F_x' . u_x' + F_y' . u_y' + F_z' . u_z')}{1 + \frac{Vu_x'}{c^2}}$$

But $u'_x = 0,\ u'_y = u',\ u'_z = 0.$

$$\therefore F_x = F'_x + \frac{V}{c^2} F'_y u'$$

By equations (4.6), (4.7) and (4.6)' the above equation becomes,

$$qE_x + qB_z u'\sqrt{1-\frac{V^2}{c^2}} = qE'_x + qu'B'_z + \frac{V}{c^2}u'\,qE'_y$$

$$\therefore E_x + B_z u'\sqrt{1-\frac{V^2}{c^2}} = E'_x + u'B'_z + \frac{V}{c^2}u'E'_y \qquad \ldots(4.9)$$

By Lorentz transformations of electric intensity $\vec{E}$,

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \therefore E'_x = E_x$$

$$\vec{E}'_{\perp} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V}\times\vec{B}\right)_{\perp}\right] \qquad \therefore E'_y = \gamma\left[E_y + \left(\vec{V}\times\vec{B}\right)_y\right]$$

$$= \gamma\,[E_y - VB_z]$$

$\therefore$ Equation (4.9) becomes,

$$E_x + B_z u'\sqrt{1-\frac{V^2}{c^2}} = E_x + u'B'_z + \frac{V}{c^2}u'\,\gamma\,(E_y - VB_z)$$

$$\therefore B_z\sqrt{1-\frac{V^2}{c^2}} = B'_z + \frac{V}{c^2}\gamma\,(E_y - VB_z)$$

$$\therefore B'_z = B_z\sqrt{1-\frac{V^2}{c^2}} - \frac{V\gamma E_y}{c^2} + \frac{V^2\gamma}{c^2}B_z$$

$$= B_z\left[\sqrt{1-\frac{V^2}{c^2}} + \frac{V^2}{c^2}\gamma\right] - \frac{V\gamma E_y}{c^2}$$

$$= B_z\left[\sqrt{1-\frac{V^2}{c^2}} + \frac{V^2}{c^2}\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\right] - \frac{V\gamma E_y}{c^2}$$

$$= B_z\left[\frac{1-\frac{V^2}{c^2}+\frac{V^2}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}}\right] - \frac{V\gamma E_y}{c^2}$$

$$= B_z\left[\frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\right] - \frac{V\gamma E_y}{c^2}$$

$$= \gamma B_z - \frac{V\gamma E_y}{c^2}$$

$$= \gamma\left[B_z - \frac{1}{c^2}(VE_y)\right]$$

$$= \gamma\left[B_z - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_z\right]$$

v) By Lorentz transformations of force, Z-component of force is given by

$$F_z' = \frac{F_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{Vu_x}{c^2}}$$

$$\therefore F_z = \frac{F_z'\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{Vu'_x}{c^2}} = F_z'\sqrt{1 - \frac{V^2}{c^2}} \qquad (\text{since } u'_x = 0)$$

By equation (4.8) and (4.8)' the above equation becomes,

$$qE_z + qVB_y - qB_xu'\sqrt{1 - \frac{V^2}{c^2}} = \sqrt{1 - \frac{V^2}{c^2}}\left[qE_z' - qu'B'_x\right]$$

$$\therefore E_z + VB_y - B_xu'\sqrt{1 - \frac{V^2}{c^2}} = \sqrt{1 - \frac{V^2}{c^2}}\, E_z' - \sqrt{1 - \frac{V^2}{c^2}}\, u'B'_x \qquad \text{...(4.10)}$$

By Lorentz transformations of electric intensity $\vec{E}$,

$$\vec{E'_\perp} = \gamma\left[\vec{E_\perp} + \left(\vec{V} \times \vec{B}\right)_\perp\right] \qquad \therefore E_z' = \gamma\left[E_z + \left(\vec{V} \times \vec{B}\right)_z\right]$$

$$= \gamma\left[E_z + VB_y\right]$$

∴ Equation (4.10) becomes

$$E_z + VB_y - B_xu'\sqrt{1 - \frac{V^2}{c^2}} = \sqrt{1 - \frac{V^2}{c^2}}\,\gamma\left[E_z + VB_y\right] - \sqrt{1 - \frac{V^2}{c^2}}\, u'B'_x$$

$$\therefore E_z + VB_y - B_xu'\sqrt{1 - \frac{V^2}{c^2}} = E_z + VB_y - u'B'_x\sqrt{1 - \frac{V^2}{c^2}}$$

$$\therefore B_x u' \sqrt{1-\frac{V^2}{c^2}} = u' B'_x \sqrt{1-\frac{V^2}{c^2}}$$

$$\therefore B'_x = B_x$$

vi) By now we have obtained the transformation equations for B'_x and B'_z in (iv) and (v). In equations (4.6), (4.7) and (4.8) there is no B'_y because the charged particle has been considered to be moving along Y'-axis relative to the frame S'. To obtain transformation equation for B'_y we shall have to consider the charged particle to be moving along Z'-axis. Then in the same manner as in(iv) we can prove that

$$B'_y = \gamma\left[B_y - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_y\right]$$

Thus from (iv), (v) and (vi) we get,

$$B'_x = B_x$$

$$B'_y = \gamma\left[B_y - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_y\right]$$

$$B'_z = \gamma\left[B_z - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_z\right]$$

The first equation shows that the magnetic induction remains the same in both the frames S and S' in the direction of relative motion between the two frames.

i.e. $\vec{B}'_{\parallel} = \vec{B}_{\parallel}$

The second and the third equations show that there is a change in the magnetic induction in the direction perpendicular to the direction of relative motion.

In general,

$$\vec{B}'_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_{\perp}\right]$$

Thus the Lorentz transformations of magnetic induction are

$$\boxed{\begin{aligned}\vec{B}'_{\parallel} &= \vec{B}_{\parallel}\\ \vec{B}'_{\perp} &= \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_{\perp}\right]\end{aligned}}$$

Discussion :

a) Let the field be purely electrical in the frame S.

i.e. $\vec{E} \neq \vec{0}$ and $\vec{B} = \vec{0}$

$$\therefore \vec{B}_{\parallel} = \vec{0}, \quad \vec{B}_{\perp} = \vec{0}$$

We shall investigate what happens in the frame S'.

$$\vec{E'}_{\parallel} = \vec{E}_{\parallel} \neq \vec{0}, \qquad \vec{E'}_{\perp} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right]$$

$$= \gamma\left[\vec{E}_{\perp} + \vec{0}\right]$$

$$= \gamma \vec{E}_{\perp} \neq 0.$$

i.e. electric field is present in the frame S'.

$$\vec{B'}_{\parallel} = \vec{B}_{\parallel} = \vec{0}, \qquad \vec{B'}_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}\right]$$

$$= \gamma\left[\vec{0} - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}\right]$$

$$= -\frac{\gamma}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}$$

$$\neq 0$$

i.e. magnetic field also is present in the frame S'.

Thus in the frame S' electric and magnetic both fields are present. Hence what appears as a purely electric field in one frame of reference is nothing but a mixture of electric and magnetic fields in another frame of reference.

b) Let the field be purely magnetic in the frame S.

i.e. $\vec{B} \neq \vec{0}$ and $\vec{E} = \vec{0}$

$$\therefore \vec{E}_{\parallel} = \vec{0}, \quad \vec{E}_{\perp} = \vec{0}$$

We shall investigate what happens in the frame S'.

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} = \vec{0}, \qquad \begin{aligned}\vec{E}'_{\perp} &= \gamma\left[\vec{E}_{\perp} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right] \\ &= \gamma\left[\vec{0} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right] \\ &= \gamma\left(\vec{V} \times \vec{B}\right)_{\perp} \\ &\neq 0.\end{aligned}$$

i.e. electric field is present in the frame S'.

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \neq \vec{0}, \qquad \begin{aligned}\vec{B}'_{\perp} &= \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}\right] \\ &= \gamma\left[\vec{B}_{\perp} - \vec{0}\right] \\ &= \gamma \vec{B}_{\perp} \\ &\neq 0\end{aligned}$$

i.e. magnetic field also is present in the frame S'.

Thus in the frame S', electric and magnetic both fields are present. Hence what appears as a purely magnetic field in one frame of reference is nothing but a mixture of electric and magnetic fields in another frame of reference.

Remark : We have seen in the second chapter (see §2.9) that space difference in one frame appears as time difference in another frame and vice versa.i.e.space and time can be converted into each other during co-ordinate transformations from one inertial frame to another inertial frame. In the third chapter we have observed that similar relationship exists between momentum and energy (see §3.9). Momentum and energy are interdependent and can be converted into each other during co-ordinate transformations. Here we observe that the same is true for electric intensity $\vec{E}$ and magnetic induction $\vec{B}$.

They also are interconnected and can be converted into each other during co-ordinate transformations.

Problem : Show that the scalar product $\vec{E} \cdot \vec{B}$ is invariant under Lorentz transformations of $\vec{E}$ and $\vec{B}$.

Solution :

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \therefore E'_x = E_x$$

$$\vec{E}_{\perp} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V} \times \vec{B}\right)_{\perp}\right] \qquad \therefore E'_y = \gamma\left[E_y + \left(\vec{V} \times \vec{B}\right)_y\right]$$

$$= \gamma\,[E_y - VB_z]$$

$$E'_z = \gamma\left[E_z + \left(\vec{V} \times \vec{B}\right)_z\right]$$

$$= \gamma\,[E_z + VB_y]$$

By Lorentz transformations of magnetic induction $\vec{B}$,

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \qquad \therefore B'_x = B_x$$

$$\vec{B}'_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}\right]$$

$$\therefore B'_y = \gamma\left[B_y - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_y\right]$$

$$= \gamma\left[B_y + \frac{1}{c^2} VE_z\right]$$

$$B'_z = \gamma\left[B_z - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_z\right]$$

$$= \gamma\left[B_z - \frac{1}{c^2} VE_y\right]$$

For the frame S',

$$\vec{E}' \cdot \vec{B}' = E_x' B'_x + E_y' B'_y + E_z' B'_z$$

$$= E_xB_x + \gamma[E_y - VB_z]\,\gamma\left[B_y + \frac{1}{c^2}VE_z\right]$$

$$+ \gamma[E_z + VB_y]\,\gamma\left[B_z - \frac{1}{c^2}VE_y\right]$$

$$= E_xB_x + \gamma^2\left[E_yB_y + \frac{V}{c^2}E_yE_z - VB_yB_z - \frac{V^2}{c^2}E_zB_z\right]$$

$$+ \gamma^2\left[E_zB_z - \frac{V}{c^2}E_yE_z + VB_yB_z - \frac{V^2}{c^2}E_yB_y\right]$$

$$= E_xB_x + \gamma^2\left[E_yB_y - \frac{V^2}{c^2}E_zB_z + E_zB_z - \frac{V^2}{c^2}E_yB_y\right]$$

$$= E_xB_x + \gamma^2\left[E_yB_y\left(1 - \frac{V^2}{c^2}\right) + E_zB_z\left(1 - \frac{V^2}{c^2}\right)\right]$$

$$= E_xB_x + \frac{1}{1 - \frac{V^2}{c^2}} \cdot \left(1 - \frac{V^2}{c^2}\right)[E_yB_y + E_zB_z]$$

$$= E_xB_x + E_yB_y + E_zB_z$$

$$= \vec{E} \cdot \vec{B}\,.$$

i.e. the scalar product $\vec{E} \cdot \vec{B}$ has the same value in the frame S and S'.

Hence it is invariant under Lorentz transformations of $\vec{E}$ and $\vec{B}$.

Note : If $\vec{E}$ and $\vec{B}$ are perpendicular to each other in the frame S, then

$$\vec{E} \cdot \vec{B} = EB\cos 90° = 0\,.$$

Since $\vec{E'} \cdot \vec{B'} = \vec{E} \cdot \vec{B}$

$$\vec{E'} \cdot \vec{B'} = 0$$

$$\therefore E'B'\cos\theta' = 0$$

Now, $E' \neq 0$ and $B' \neq 0$. Hence, $\cos\theta' = 0$. $\therefore \theta' = 90°$.

∴ $\vec{E'}$ and $\vec{B'}$ are perpendicular to each other. i.e. $\vec{E}$ and $\vec{B}$ are perpendicular to each other in the frame S' also.

Thus if $\vec{E}$ and $\vec{B}$ are perpendicular to each other in one inertial frame then they are perpendicular to each other in all inertial frames.

Problem : Show that the quantity $E^2 - c^2B^2$ remains invariant under Lorentz transformations of $\vec{E}$ and $\vec{B}$.

Solution :

As we have done in the previous problem we can get the following equations :

$$E'_x = E_x, \qquad E'_y = \gamma\,[E_y - VB_z], \qquad E'_z = \gamma\,[E_z + VB_y]$$

$$B'_x = B_x, \qquad B'_y = \gamma\left[B_y + \frac{V}{c^2}E_z\right], \qquad B'_z = \gamma\left[B_z - \frac{V}{c^2}E_y\right]$$

For the frame S'

$$E'^2 - c^2B'^2$$

$$= (E'_x)^2 + (E'_y)^2 + (E'_z)^2 - c^2(B'_x)^2 - c^2(B'_y)^2 - c^2(B'_z)^2$$

$$= E_x^2 + \gamma^2[E_y - VB_z]^2 + \gamma^2[E_z + VB_y]^2 - c^2B_x^2$$

$$- c^2\gamma^2\left[B_y + \frac{V}{c^2}E_z\right]^2 - c^2\gamma^2\left[B_z - \frac{V}{c^2}E_y\right]^2$$

$$= E_x^2 + \gamma^2\left[E_y^2 - 2VE_yB_z + V^2B_z^2 + E_z^2 + 2VE_zB_y + V^2B_y^2\right]$$

$$- c^2B_x^2 - c^2\gamma^2\left[B_y^2 + \frac{2V}{c^2}E_zB_y + \frac{V^2}{c^4}E_z^2 + B_z^2 - \frac{2V}{c^2}E_yB_z + \frac{V^2}{c^4}E_y^2\right]$$

$$= E_x^2 + \gamma^2\Big[E_y^2 - 2VE_yB_z + V^2B_z^2 + E_z^2 + 2VE_zB_y + V^2B_y^2 - c^2B_y^2$$

$$- 2VE_zB_y - \frac{V^2}{c^2}E_z^2 - c^2B_z^2 + 2VE_yB_z - \frac{V^2}{c^2}E_y^2\Big] - c^2B_x^2$$

$$= E_x^2 + \gamma^2\left[E_y^2 + V^2B_z^2 + E_z^2 + V^2B_y^2 - c^2B_y^2 - \frac{V^2}{c^2}E_z^2 - c^2B_z^2 - \frac{V^2}{c^2}E_y^2\right] - c^2B_x^2$$

$$= E_x^2 + \gamma^2 \left[E_y^2 \left(1 - \frac{V^2}{c^2}\right) + E_z^2 \left(1 - \frac{V^2}{c^2}\right) + B_z^2 (V^2 - c^2) + B_y^2 (V^2 - c^2) \right]$$

$$- c^2 B_x^2$$

$$= E_x^2 + \frac{1}{1 - \frac{V^2}{c^2}} \left[\left(1 - \frac{V^2}{c^2}\right) \left(E_y^2 + E_z^2\right) + (V^2 - c^2) \left(B_y^2 + B_z^2\right) \right] - c^2 B_x^2$$

$$= E_x^2 + \left(E_y^2 + E_z^2\right) + \frac{V^2 - c^2}{c^2 - V^2} \cdot c^2 \left(B_y^2 + B_z^2\right) - c^2 B_x^2$$

$$= \left(E_x^2 + E_y^2 + E_z^2\right) - c^2 \left(B_y^2 + B_z^2\right) - c^2 B_x^2$$

$$= E^2 - c^2 \left(B_x^2 + B_y^2 + B_z^2\right)$$

$$= E^2 - c^2 B^2$$

i.e. $E^2 - c^2 B^2$ has the same value in the frame S and S'. Hence it is invariant under Lorentz transformations of $\vec{E}$ and $\vec{B}$.

Remark : This invariance is analogous to the invariance of (i) $x^2 + y^2 + z^2 - c^2 t^2$ under Lorentz transformations of space and time co-ordinates and (ii) $p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$ under Lorentz transformations of momentum and energy.

4.4 FIELD INTENSITIES OF AN UNIFORMLY MOVING POINT CHARGE:

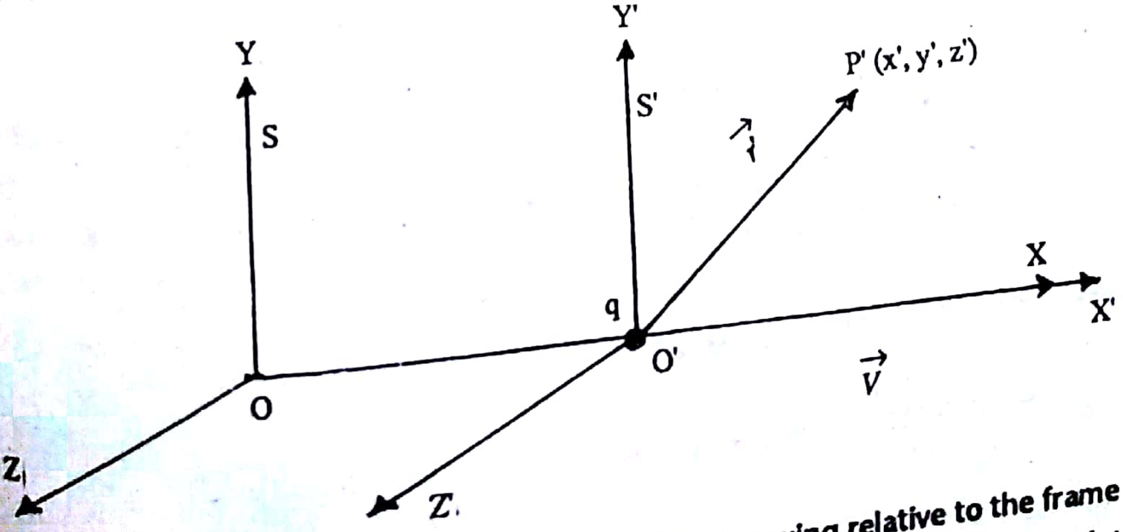


Fig.4.1 A point charge q stationary in the frame S' but moving relative to the frame S.

Let us consider a point charge q lying at the origin O' of the frame S'. It is stationary in the frame S' and hence it produces only intensity of the electric field

$$= E_x^2 + \gamma^2\left[E_y^2\left(1-\frac{V^2}{c^2}\right)+E_z^2\left(1-\frac{V^2}{c^2}\right)+B_z^2(V^2-c^2)+B_y^2(V^2-c^2)\right]$$

$$-c^2B_x^2$$

$$= E_x^2 + \frac{1}{1-\frac{V^2}{c^2}}\left[\left(1-\frac{V^2}{c^2}\right)\left(E_y^2+E_z^2\right)+(V^2-c^2)\left(B_y^2+B_z^2\right)\right]-c^2B_x^2$$

$$= E_x^2 + \left(E_y^2+E_z^2\right)+\frac{V^2-c^2}{c^2-V^2}\cdot c^2\left(B_y^2+B_z^2\right)-c^2B_x^2$$

$$= \left(E_x^2+E_y^2+E_z^2\right)-c^2\left(B_y^2+B_z^2\right)-c^2B_x^2$$

$$= E^2-c^2\left(B_x^2+B_y^2+B_z^2\right)$$

$$= E^2-c^2B^2$$

i.e. $E^2 - c^2B^2$ has the same value in the frame S and S'. Hence it is invariant under Lorentz transformations of $\vec{E}$ and $\vec{B}$.

Remark : This invariance is analogous to the invariance of (i) $x^2 + y^2 + z^2 - c^2t^2$ under Lorentz transformations of space and time co-ordinates and (ii) $p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$ under Lorentz transformations of momentum and energy.

4.4 FIELD INTENSITIES OF AN UNIFORMLY MOVING POINT CHARGE:

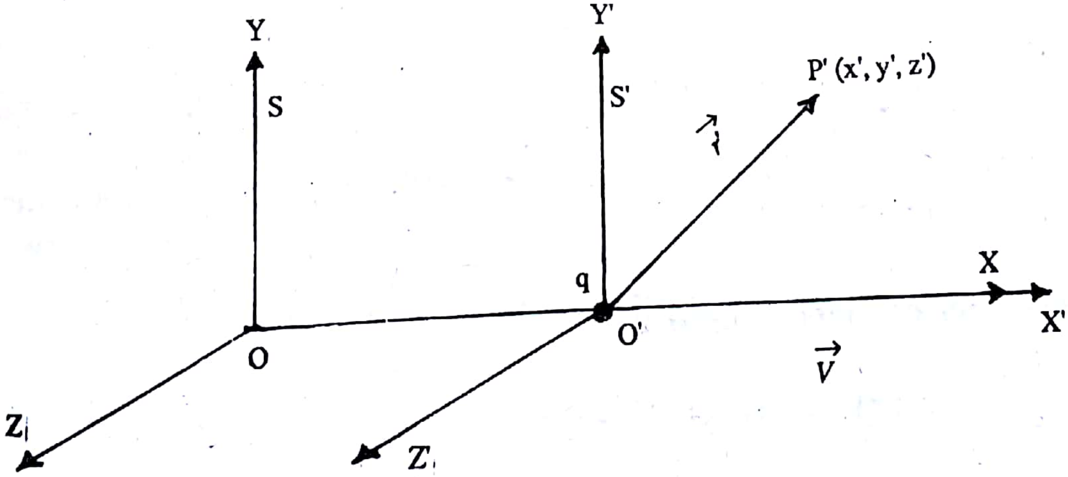


Fig.4.1 A point charge q stationary in the frame S' but moving relative to the frame S.

Let us consider a point charge q lying at the origin O' of the frame S'. It is stationary in the frame S' and hence it produces only static electric field in the frame S'. The intensity of the electric field at a point P' having co-ordinates (x', y', z') and position vector $\vec{r'}$ relative to the frame S' is given by

$$\vec{E'} = \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\cdot\frac{\vec{r'}}{r'} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r'^3}\vec{r'}$$

$$\text{Now, } \vec{r'} = x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$$

$$r'^2 = x'^2 + y'^2 + z'^2$$

$$\therefore r' = (x'^2 + y'^2 + z'^2)^{1/2}$$

$$\therefore \vec{E'} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}\left(x'\vec{i'} + y'\vec{j'} + z'\vec{k'}\right)$$

$$\therefore E_x' = \frac{1}{4\pi\epsilon_0}\frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}x'$$

$$E_y' = \frac{1}{4\pi\epsilon_0}\frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}y'$$

$$E_z' = \frac{1}{4\pi\epsilon_0}\frac{q}{(x'^2 + y'^2 + z'^2)^{3/2}}z'$$

Since the charge is stationary in the frame S', it does not give rise to magnetic field.

Hence $\vec{B'} = \vec{0}$. $\therefore B_x' = 0,\ B_y' = 0,\ B_z' = 0.$

Now, since the frame S' is moving relative to the frame S with velocity V along X-axis, the charge is moving with velocity V along X-axis relative to the frame S. We shall find out the intensities of electric and magnetic fields of the charge q relative to the frame S in what follows :

Intensity of electric field :

By Lorentz transformations of electric intensity $\vec{E}$,

$$\vec{E'_{\parallel}} = \vec{E}_{\parallel}, \qquad \vec{E'_{\perp}} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V}\times\vec{B}\right)_{\perp}\right]$$

Taking inverse transformations,

$$\vec{E}_{\parallel} = \vec{E}'_{\parallel} \qquad \vec{E}_{\perp} = \gamma\left[\vec{E}'_{\perp} - \left(\vec{V} \times \vec{B}'\right)_{\perp}\right]$$

But $\vec{B}' = \vec{0}$

$$\therefore \vec{E}_{\perp} = \gamma \vec{E}'_{\perp}$$

i) $E_x = E'_x$

$$= \frac{1}{4\pi\epsilon_0} \frac{q\, x'}{(x'^2 + y^2 + z'^2)^{3/2}}$$

By Lorentz transformations of space and time co-ordinates,

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma(x - Vt), \qquad y' = y, \qquad z' = z$$

$$\therefore x'^2 + y'^2 + z'^2 = \gamma^2 (x - Vt)^2 + y^2 + z^2$$

$$\therefore E_x = \frac{1}{4\pi\epsilon_0} \frac{q\,\gamma(x - Vt)}{[\gamma^2 (x - Vt)^2 + y^2 + z^2]^{3/2}}$$

ii) $E_y = \gamma E'_y$

$$= \gamma \cdot \frac{1}{4\pi\epsilon_0} \frac{qy'}{(x'^2 + y'^2 + z'^2)^{3/2}}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q\gamma y}{[\gamma^2 (x - Vt)^2 + y^2 + z^2]^{3/2}}$$

iii) $E_z = \gamma E'_z$

$$= \gamma \cdot \frac{1}{4\pi\epsilon_0} \frac{qz'}{(x'^2 + y'^2 + z'^2)^{3/2}}$$

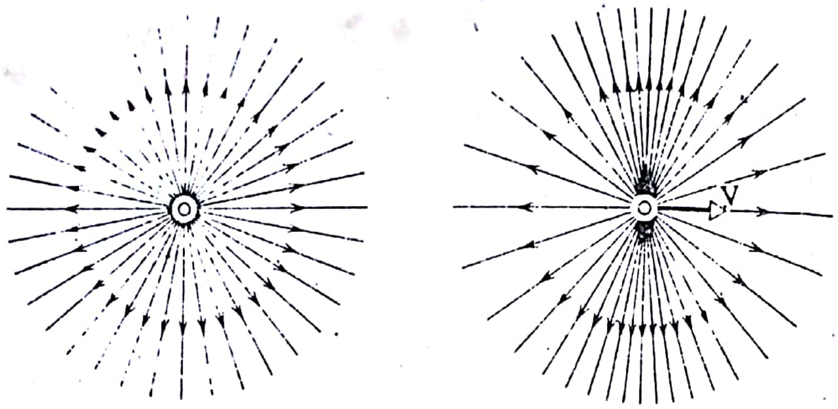
$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q\gamma z}{[\gamma^2(x - Vt)^2 + y^2 + z^2]^{3/2}}$$

Now,

$$\vec{E} = E_x \vec{i} + E_y \vec{j} + E_z \vec{k}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q\gamma}{[\gamma^2 (x - Vt)^2 + y^2 + z^2]^{3/2}} \left[(x - Vt)\,\vec{i} + y\vec{j} + z\vec{k}\right] \qquad \text{...(4.11)}$$

For the observer in the frame S', the electric field is radial, inverse square and spherically symmetric as shown in Fig. 4.2 (a). We shall investigate how the electric field appears in the frame S relative to which the charge is moving with velocity V along X-axis.



(a) (b)

Fig. 4.2 (a) The electric field is spherically symmetric in the frame S' where the charge is stationary. (b) The electric field is not spherically symmetric in the frame S in which the charge is moving with velocity V along X - axis

We shall consider the electric field of the charge at the instant $t = 0$. At this instant the charge lies at the origin O momentarily. At other instants of time the field will appear to be the same but translated to the right by a distance Vt. At $t = 0$, the equation (4.11) becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{[\gamma^2 x^2 + y^2 + z^2]^{3/2}} \left[x\vec{i} + y\vec{j} + z\vec{k} \right]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q\gamma}{[\gamma^2 x^2 + y^2 + z^2]^{3/2}} \vec{r} \qquad \text{...(4.12)}$$

Let the angle made by $\vec{r}$ with X-axis be θ.

$$\therefore x = r\cos\theta$$

Now, $r^2 = x^2 + y^2 + z^2$

$$\therefore r^2 = r^2\cos^2\theta + y^2 + z^2$$

$$\therefore y^2 + z^2 = r^2(1 - \cos^2\theta) = r^2\sin^2\theta$$

$$\therefore \gamma^2 x^2 + y^2 + z^2 = \frac{1}{1 - \frac{V^2}{c^2}} \cdot (r\cos\theta)^2 + r^2\sin^2\theta$$

Let $\frac{V}{c} = \beta$

$$\therefore \gamma^2 x^2 + y^2 + z^2 = \frac{r^2 \cos^2\theta}{1-\beta^2} + r^2 \sin^2\theta$$

$$= \frac{r^2 \cos^2\theta + r^2 \sin^2\theta - \beta^2 r^2 \sin^2\theta}{1-\beta^2}$$

$$= \frac{r^2 (\sin^2\theta + \cos^2\theta) - \beta^2 r^2 \sin^2\theta}{1-\beta^2}$$

$$= \frac{r^2 - \beta^2 r^2 \sin^2\theta}{1-\beta^2}$$

$$= \frac{r^2 (1 - \beta^2 \sin^2\theta)}{1-\beta^2}$$

Hence equation (4.12) becomes,

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q\gamma (1-\beta^2)^{3/2}}{[r^2 (1-\beta^2 \sin^2\theta)]^{3/2}} \vec{r}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{1-\beta^2}} \frac{q (1-\beta^2)^{3/2}}{r^3 (1-\beta^2 \sin^2\theta)^{3/2}} \vec{r}$$

$$= \frac{1}{4\pi\epsilon_0} \cdot \frac{q (1-\beta^2)}{r^2 (1-\beta^2 \sin^2\theta)^{3/2}} \frac{\vec{r}}{r}$$

VIMP

You can see here, $\vec{E}$ varies inversely as r^2 and hence the electric field is inverse square. It is radial. But $\vec{E}$ varies with respect to θ sinusoidally. Hence the field is not spherically symmetric. This is shown in Fig. 4.2 (b).

Intensity of Magnetic field :

By Lorentz transformations of magnetic induction $\vec{B}$,

$$\vec{B'}_{\parallel} = \vec{B}_{\parallel}, \qquad \vec{B'}_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V} \times \vec{E}\right)_{\perp}\right]$$

Taking inverse transformations,

$$\vec{B}_{\parallel} = \vec{B'}_{\parallel}, \qquad \vec{B}_{\perp} = \gamma\left[\vec{B'}_{\perp} + \frac{1}{c^2}\left(\vec{V} \times \vec{E'}\right)_{\perp}\right]$$

But $\vec{B'} = \vec{0}$ $\qquad \therefore \vec{B'_{\parallel}} = \vec{0}, \quad \vec{B'_{\perp}} = \vec{0}$

$$\therefore \vec{B_{\parallel}} = \vec{0} \text{ and } \vec{B_{\perp}} = \frac{\gamma}{c^2}\left(\vec{V} \times \vec{E'}\right)_{\perp}$$

iv) $B_x = 0$

v) $B_y = \frac{\gamma}{c^2}\left(\vec{V} \times \vec{E'}\right)_y = \frac{\gamma}{c^2}(0 - VE'_z) = -\frac{\gamma V E'_z}{c^2}$

In part (iii) we have proved

$E_z = \gamma E'_z \qquad \therefore E'_z = \frac{1}{\gamma} E_z$

$$\therefore B_y = -\frac{\gamma V}{c^2} \cdot \frac{1}{\gamma} E_z = \frac{-VE_z}{c^2} = \frac{\left(\vec{V} \times \vec{E}\right)_y}{c^2}$$

vi) $B_z = \frac{\gamma}{c^2}\left(\vec{V} \times \vec{E'}\right)_z = \frac{\gamma}{c^2}(VE'_y)$

In part (ii) we have proved

$E_y = \gamma E'_y \qquad \therefore E'_y = \frac{1}{\gamma} E_y$

$$\therefore B_z = \frac{\gamma}{c^2}\left(\frac{V}{\gamma} E_y\right) = \frac{VE_y}{c^2} = \frac{\left(\vec{V} \times \vec{E}\right)_z}{c^2}$$

From (iv), (v) and (vi) we get,

$$B_x = 0, \qquad B_y = \frac{\left(\vec{V} \times \vec{E}\right)_y}{c^2}, \qquad B_z = \frac{\left(\vec{V} \times \vec{E}\right)_z}{c^2}$$

The first equation can be written as $\vec{B_{\parallel}} = \vec{0}$.

The second and the third equation can be written in general as

$$\vec{B'_{\perp}} = \frac{\left(\vec{V} \times \vec{E}\right)_{\perp}}{c^2}$$

.5 FORCES AND FIELDS NEAR A CURRENT - CARRYING WIRE :

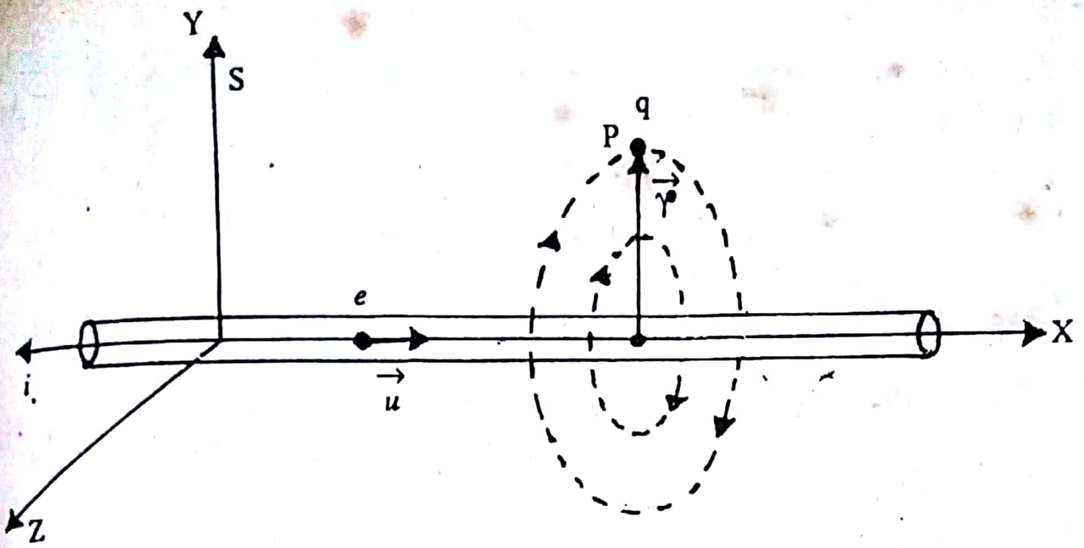


Fig. 4.3 A current - carrying wire lying along x-axis in the frame S.

A long straight wire carrying a current i is lying along X-axis in the frame S. Then electrons are moving inside the wire with a drift velocity $\vec{u}$ along X-axis (Fig. 4.3).

Let the linear density of the charge (i.e. charge per unit length of the wire) due to electrons be λ^-. Hence the current in the wire is given by

$$i = \frac{q}{t} = \frac{\lambda^- . l}{t} = (\lambda^-) . \left(\frac{l}{t}\right) = (\lambda^-)\, u$$

Its direction is opposite to that of $\vec{u}$ i.e. along negative X-axis.

Let the linear density of the charge due to positive ions of the wire, which remain at rest in the frame S be λ^+.

Field relative to the frame S :

Now, since λ^- and λ^+ are equal in magnitude, $\lambda^- + \lambda^+ = 0$. Hence there is no net charge in the wire and the intensity of the electric field in the frame S is zero. i.e. $\vec{E} = \vec{0}$.

$$\therefore E_x = 0, \qquad E_y = 0, \qquad E_z = 0$$

The intensity of the magnetic field produced around the wire is given by

$$B = \frac{\mu_0}{4\pi} . \frac{2i}{r} = \frac{\mu_0 (\lambda^-)\, u}{2\pi r}$$

It is directed along the concentric circles around the wire, with the wire as the axis (Fig. 4.3). Its direction can be found out using the right hand rule.

The components of the magnetic induction at P along X, Y, and Z-axes are as follows :

$$B_x = 0, \qquad B_y = 0, \qquad B_z = -\frac{\mu_0 (\lambda^-) u}{2\pi r}$$

The negative sign in Z-component indicates that the direction of the magnetic induction $\vec{B}$ at P is opposite to that of Z-axis.

Force acting relative to frame S :

A test charge q is kept at a distance r from the wire. (Fig. 4.3). Force exerted by the electric field on the test charge is given by

$$\vec{F}_E = q\vec{E} = \vec{0} \qquad \left(\because \vec{E} = \vec{0}\right)$$

Force exerted by the magnetic field is given by

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Now, the charge is stationary at P relative to the frame S.

i.e $\vec{v} = \vec{0}$.

$$\therefore \vec{F}_B = \vec{0}$$

Thus no electromagnetic force acts on the test charge relative to the frame S.

Field relative to frame S' :

We shall investigate what happens in the frame S'. Let the velocity of the frame S' relative to S along X-axis be

$$V = u.$$

By Lorentz transformations of electric intensity $\vec{E}$,

$$\vec{E'_{\parallel}} = \vec{E}_{\parallel},$$

$$\vec{E'}_{\perp} = \gamma\left[\vec{E}_{\perp} + \left(\vec{V}\times\vec{B}\right)_{\perp}\right]$$

$$= \gamma\left[\vec{0} + \left(\vec{u}\times\vec{B}\right)_{\perp}\right]$$

$$= \gamma\left(\vec{u}\times\vec{B}\right)_{\perp}$$

Now $$\vec{u}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u & 0 & 0 \\ 0 & 0 & B_z \end{vmatrix}$$

i) $E'_x = E_x = 0$

ii) $$E'_y = \gamma\left(\vec{u}\times\vec{B}\right)_y$$

$$= \gamma\,(0 - uB_z) = -\gamma\,uB_z = -\gamma\,u\left(\frac{-\mu_0(\lambda^-)u}{2\pi r}\right) = \frac{\mu_0\gamma(\lambda^-)u^2}{2\pi r}$$

iii) $$E'_z = \gamma\left(\vec{u}\times\vec{B}\right)_z = \gamma\times 0 = 0$$

By Lorentz transformations of magnetic induction $\vec{B}$

$$\vec{B'}_{\parallel} = \vec{B}_{\parallel}, \qquad \vec{B'}_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{V}\times\vec{E}\right)_{\perp}\right]$$

$$= \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\left(\vec{u}\times\vec{0}\right)_{\perp}\right]$$

$$= \gamma\vec{B}_{\perp}$$

iv) $B'_x = B_x = 0$

v) $B'_y = \gamma\,B_y = \gamma\times 0 = 0$

vi) $$B'_z = \gamma\,B_z = \gamma\left[-\frac{\mu_0(\lambda^-)u}{2\pi r}\right] = -\frac{\mu_0\gamma(\lambda^-)u}{2\pi r}$$

Thus in the frame S' there is an electric field along Y'-axis and magnetic field along negative Z'-axis.

Force acting relative to the frame S' :

Force exerted by the electric field on the test charge is given by

$$\vec{F'_E} = q\,\vec{E'}$$

$$\therefore\; F'_E = qE'_y = \frac{\mu_0\,\gamma(\lambda^-)\,q\,u^2}{2\pi r} \qquad ...(4.13)$$

This force is directed along the positive Y'-axis.

Force exerted by magnetic field is given by

$$\vec{F'_B} = q\vec{v} \times \vec{B'}$$

The test charge is stationary relative to the frame S. Hence it is moving with velocity $V = u$ along the negative X'-axis relative to the frame S'.

$$\therefore\; \vec{F'_B} = q\left(\vec{u} \times \vec{B'_z}\right)$$

$$\therefore\; F'_B = quB'_z \sin 90^\circ$$

$$= qu\,\frac{\mu_0\,\gamma(\lambda^-)\,u}{2\pi r}$$

$$= \frac{\mu_0\gamma(\lambda^-)qu^2}{2\pi r} \qquad .....(4.14)$$

Since $\vec{u}$ is directed along negative X'-axis and $\vec{B'_z}$ is directed along negative Z'-axis, the above force is directed along the negative Y'-axis (by using right hand rule).

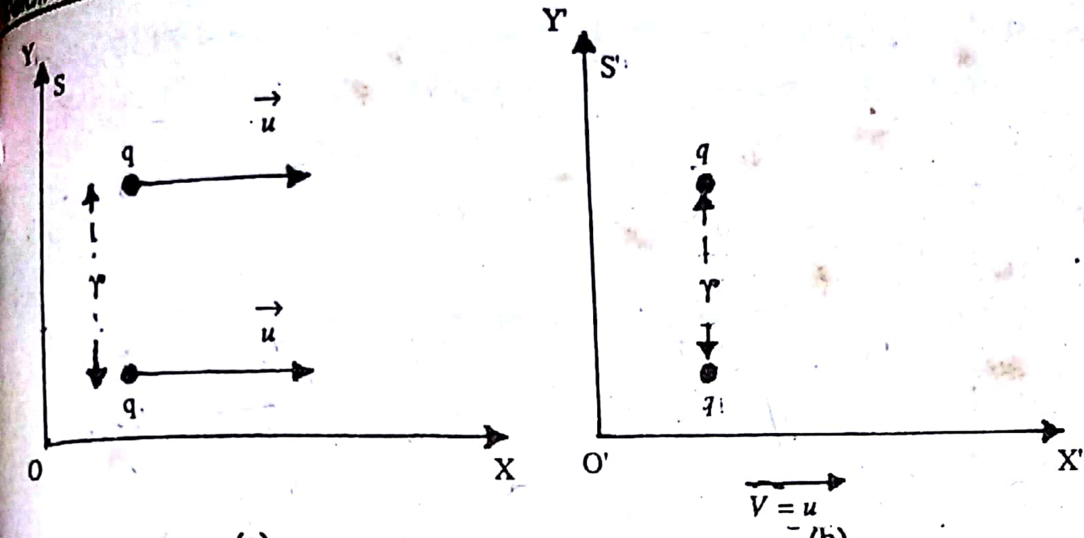
From equations (4.13) and (4.14) we see that

$$F'_E = F'_B$$

i.e. the forces exerted by the electric and magnetic fields are equal in magnitude. But they are opposite in direction. Hence the total electromagnetic force acting on the test charge is zero in the frame S' also.

4.6 FORCES BETWEEN MOVING CHARGES :

Let us consider two equal charges q separated by a distance r, lying in X-Y place and having the same X-co-ordinate, relative to the frame S. They are moving with the same velocity u along X-axis relative to the frame S.



(a) (b)

Fig. 4.4(a) Two equal charges are moving with the same velocity u along X-axis relative to the frame S. (b) The same charges appear to be at rest relative to the frame S'.

The charges exert repulsive electric forces on each other and these forces are directed along Y-axis.

Each moving charge is equivalent to a current and it gives rise to a magnetic field $\vec{B}$ around the path of the charge. It is directed along the concentric circles around the path with the path as the axis. Its direction can be found out using the right hand rule. The other charge is moving in the magnetic field $\vec{B}$ due to the first charge. Hence a magnetic force $\vec{F}$ acts on the second charge and is given by

$$\vec{F} = q\,\vec{u} \times \vec{B}$$

This force acting on each charge is directed towards the other charge i.e. these forces are attractive in nature and are directed along Y-axis.

We shall find out the resultant of the electric and magnetic forces. For this purpose let us introduce the frame S' moving with velocity $V = u$ relative to the frame S along x-axis. Now in this frame S', the charges are stationary and hence they do not give rise to any magnetic field. Thus they do not exert any magnetic force on each other. The electric force of repulsion exerted by the two charges on each other is given by

$$F'_x = 0, \qquad F'_y = \frac{1}{4\pi\epsilon_0}\,\frac{q^2}{r^2}, \qquad F'_z = 0$$

We shall transform these equations to the frame S as follows :

By Lorentz transformations of space and time co-ordinates,

$$y' = y \qquad \therefore r' = r$$

$$\therefore F_y' = \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{r^2} \qquad \ldots(4.15)$$

i) By Lorentz transformations of force,

$$F_x' = \frac{F_x - \frac{V}{c^2}\left(\vec{F} \cdot \vec{u}\right)}{1 - \frac{Vu_x}{c^2}}$$

Taking inverse transformations,

$$F_x = \frac{F_x' + \frac{V}{c^2}\left(\vec{F'} \cdot \vec{u'}\right)}{1 + \frac{Vu'_x}{c^2}}$$

Now, $\vec{u'} = \vec{0}$, and $u'_x = 0$ for the charges.

$$\therefore F_x = F_x' = 0$$

ii) $$F_y' = \frac{F_y\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V \cdot u_x}{c^2}}$$

$$\therefore F_y = \frac{F_y'\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V \cdot u'_x}{c^2}}$$

$$= F_y'\sqrt{1 - \frac{u^2}{c^2}} \qquad (\because V = u)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} \sqrt{1 - \frac{u^2}{c^2}}$$ by equation (4.15)

iii) $$F_z' = \frac{F_z\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V \cdot u_x}{c^2}}$$

$$\therefore F_z = \frac{F'_z\sqrt{1-\frac{V^2}{c^2}}}{1+\frac{V u'_x}{c^2}} = F'_z\sqrt{1-\frac{u^2}{c^2}} = 0$$

Hence the resultant force exerted by the two charges on each other relative to the frame S is given by

$$F_y = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\sqrt{1-\frac{u^2}{c^2}}$$

When $u = 0$, $F_y = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}$ which is the electric force exerted by one charge on the other according to Coulomb's law. i.e. if the charges are stationary they exert only the electric force. When they are moving with a velocity u, the resultant force exerted by them on each other is given by

$$F_y = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}\sqrt{1-\frac{u^2}{c^2}} < \frac{1}{4\pi\epsilon_0} \cdot \frac{q^2}{r^2}$$

i.e. the resultant force is slightly less than the electric force. This happens due to the magnetic force exerted by the moving charges on each other, which is attractive in nature whereas the electric force is repulsive in nature.

Thus we observe that the effect of magnetic force is much less than the electric force. The magnetic force depends upon the velocity u of the charges. Hence the magnetic force is relativistic in nature. Thus the electric force exerted by the two charges on each other is greater than the magnetic force.

QUESTIONS

1. State the equation representing the Lorentz force law for a charged particle moving in an electromagnetic field.
2. Using Lorentz transformations of force derive transformation equations for electric intensity $\vec{E}$ and magnetic induction $\vec{B}$ between two inertial frames in uniform relative translational motion along their common X-axis.
[B.U. May 1989, Oct. 1989, May 1990, Oct. 1990, April 1991, April 1992, May 1994]
3. Obtain the components of electric and magnetic field vectors of an uniformly moving point electric charge in an inertial frame of reference. [B.U. April 1991, April 1992]
4. Show that the electric field of a uniformly moving point electric charge in an inertial frame of reference loses its spherical symmetry. [B.U. Oct. 1991]
5. A long straight wire is carrying a current and lies along X-axis of the frame S. Find the components of electric intensity and magnetic induction produced by the wire relative to the frame S and S'. Hence show that if a stationary charge is lying in the field, near the conductor then no electromagnetic force acts on it relative to the frame S as well as S'.
6. Two equal charges are moving with the same velocity along X-axis. Using Lorentz transformations of force find the resultant force exerted by the two charges on each other. Hence show that the electric force exerted by the two charges on each other is greater than the magnetic force.

7. Obtain expressions for the forces between two identical electric point charges moving with equal velocity in an inertial frame of reference S. Discuss the significance of the result. [B.U. April 1991, Oct. 1991]

PROBLEMS

1. Show that the scalar product $\vec{E}\cdot\vec{B}$ is unchanged by a Lorentz transformation of $\vec{E}$ and $\vec{B}$. [B.U. Oct. 1989, May 1994]
2. If electric and magnetic fields are perpendicular to one another in one inertial frame then prove that they are perpendicular in all inertial frames.[B.U. May 1989]
3. Show that the quantity $E^2 - c^2 B^2$ is invariant under Lorentz transformations of electric intensity $\vec{E}$ and magnetic induction $\vec{B}$. [B.U. May 1990]
4. If $\vec{E}$ and $\vec{B}$ are electric intensity and magnetic induction respectively then prove that if $E > cB$ in one inertial frame S then $E' > cB'$ in any other inertial frame S'.

FOOD FOR THOUGHT

1. Special theory of Relativity has led to a formal clarification of Maxwell's equations of the electromagnetic field, in particular it has led to an understanding of the essential oneness of the electric and the magnetic field.

   - Albert Einstein

* * * * *

2. What led me more or less directly to the special theory of relativity was the conviction that the electromotive force acting on a body in motion in a magnetic field was nothing else but an electric field. *- Albert Einstein*

* * * * *

3. The heuristic method of the special theory of relativity is characterized by the following principle : only those equations are admissible as an expression of natural laws which do not change their form when the co-ordinates are changed by means of Lorentz transformations (covariance of equations with respect to Lorentz transformations).

   This method led to the discovery of the necessary connection between momentum and energy, between electric and magnetic field strength, electrostatic and electrodynamic forces, inert mass and energy; and the number of independent concepts and fundamental equations in physics was thereby reduced.

   - Albert Einstein

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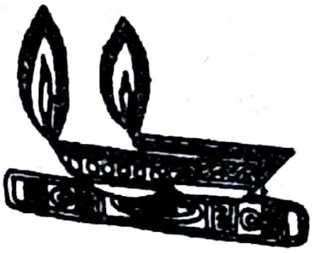


5

The Geometric Representation of Space - Time

5.1 MINKOWSKI'S FOUR-DIMENSIONAL WORLD :

As we have discussed in § 2.9 of the second chapter, according to the Special theory of Relativity, space and time are intimately and cohesively connected with each other. They can be converted into each other during co-ordinate transformations from one inertial frame to another.

V.IMP

In 1908, H. Minkowski suggested that the world of physical phenomena is four dimensional and the events occurring in this world are described by four co-ordinates (x, y, z, t). The time co-ordinate t is not independent of space co-ordinates (x, y, z). It is connected and equivalent to the space co-ordinates. It plays the same role as space co-ordinates. Even the space co-ordinates depend on time co-ordinate. Such four dimensional space is called Minkowski's space - time continuum or Minkowski world.

A physical event becomes a point in Minkowski world, **called world point**. The trajectory of a moving particle is called **world line**. A vector of Minkowski world is called **world vector** or **four-vector** in order to distinguish it from the ordinary three dimensional vectors.

Radius four-vector :

In Euclidean space of Newtonian Relativity the length (r) of the radius vector is given by

$$r^2 = x^2 + y^2 + z^2$$

In § 1.7 of the first chapter we have proved that this distance between the origin and the point (x, y, z) remains invariant in all inertial frames of reference under Galilean transformations of space-time co-ordinates.

In a problem immediately following § 2.11 of the second chapter we have proved that the quantity

$$x^2 + y^2 + z^2 - c^2t^2$$

remains invariant in all inertial frames of reference under Lorentz transformations of space-time co-ordinates. Hence in Minkowski world the length (s) of the radius four-vector is defined as

$$s^2 = x^2 + y^2 + z^2 - c^2t^2$$

$$= x^2 + y^2 + z^2 + i^2 \ c^2t^2 \qquad \text{where} \quad i = \sqrt{-1}$$

$$s^2 = x^2 + y^2 + z^2 + (i\,ct)^2$$

Generally the co-ordinates of a world point are denoted by (x_1, x_2, x_3, x_4) such that

$$x_1 = x, \qquad x_2 = y, \qquad x_3 = z, \qquad x_4 = ict$$

$$\therefore s^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$$

(A vector having components (x_1, x_2, x_3, x_4) is called **radius four-vector.**) Since s^2 remains invariant under Lorentz transformations, the Lorentz transformations are nothing but the rotation of the co-ordinate axes, X_1, X_2, X_3, X_4 in space-time or in Minkowski world.

Momentum four-vector :

In a problem immediately following § 3.10 of the third chapter we have proved that the quantity

$$p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2}$$

remains invariant in all inertial frames under Lorentz transformations of energy and momentum. Hence a vector having components $\left(p_x, p_y, p_z, \frac{iE}{c}\right)$ is called **momentum four-vector.**

First three components of a four-vector are called **space components** and the fourth component is called **time component** of the four-vector. Generally the **time components** of all four-vectors are imaginary.

Remark : You can understand easily, the Special theory of Relativity leads to the four dimensional formulation of Physics. When the laws of physics are expressed in four dimensional covariant form, we have deeper understanding of the natural laws and a clear picture of all physical quantities and their relationships. It has led to the discovery of new phenomena and relationships.

5.2 SPACE-TIME DIAGRAMS :

In 1908, Minkowski introduced the use of graphs in Special Relativity in order to describe the motion of a particle in space-time continuum and to reveal various aspects of the space-time relationships. Such graphs are called Minkowski space-time diagrams.

In a space-time diagram only one space axis, the X-axis, is considered ignoring Y and Z-axes. The reason is that during Lorentz transformations, only the X-co-ordinate changes while Y and Z-co-ordinates remain the same. The X-axis is kept horizontal and the time axis is kept vertical (Fig. 5.1). In order that the units and dimensions of the two co-ordinates would remain the same, time t is multiplied by c, the velocity of light which is an universal constant remaining the same in all inertial frames of reference. The quantity $w = ct$ will have the same unit and dimensions as those of x. It is plotted on the vertical axis.

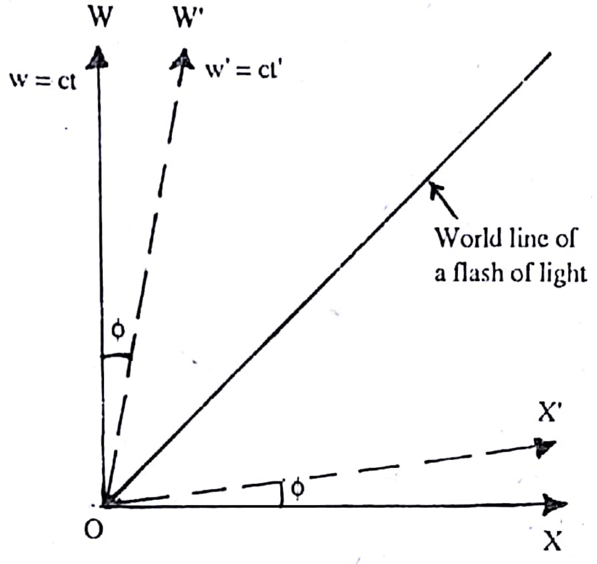


Fig. 5.1 Minkowski space--time diagram

An event occurring in space-time has co-ordinates (x, t) and is represented by a point in the space-time diagram.

For a flash of light starting from the origin of the frame S at $t = 0$,

$$c = \frac{x}{t} \quad \text{i.e.} \quad x = ct \quad \text{or} \quad x = w$$

Hence the world line of the flash of light is the bisector of the angle between the space and time axes. (Fig. 5.1). This is true for all inertial frames of reference.

By Lorentz transformations of space and time co-ordinates,

$$x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{x - \frac{V}{c}(ct)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Let $\frac{V}{c} = \beta$ and $ct = w$

$$\therefore x' = \frac{x - \beta w}{\sqrt{1 - \beta^2}}$$

Moreover, $$t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore ct' = \frac{ct - \frac{V}{c}x}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore w' = \frac{w - \beta x}{\sqrt{1 - \beta^2}}$$

Thus we have

$$x' = \frac{x - \beta w}{\sqrt{1 - \beta^2}}, \qquad w' = \frac{w - \beta x}{\sqrt{1 - \beta^2}}$$

In this form, the equations are very symmetric.

For the frame S, $x = 0$ gives the time axis of the space-time diagram and $w = 0$ gives its space axis. In the same manner, for the frame S', $x' = 0$ would give the time axis and $w' = 0$ would give the space axis.

Let $x' = 0$ $\qquad \therefore x - \beta w = 0$

$$\therefore x = \beta w = \frac{V}{c} . ct = Vt$$

Thus $x = Vt$ is the equation of W'-axis. Hence W'-axis starts from the origin O and is inclined to W-axis at an angle ϕ (Fig. 5.1) such that

$$\tan \phi = \frac{x}{w} = \beta = \frac{V}{c}$$

But $\frac{V}{c} < 1$. Hence $\tan \phi < 1$ and $\phi < 45°$.

Now let $w' = 0$ $\qquad \therefore w - \beta x = 0$

$$\therefore w = \beta x = \frac{V}{c}x$$

Thus $w = \frac{V}{c}x$ is the equation of X'-axis. Even X'-axis starts from the origin O and is inclined to X-axis at an angle ϕ such that

$$\tan\phi = \frac{w}{x} = \beta = \frac{V}{c}$$

Hence X'-axis and W'-axis are equally inclined to the X-axis and W-axis respectively. The angle of inclination ϕ depends upon the velocity V of the frame S' relative to the frame S.

5.3 CALIBRATION OF SPACE AND TIME AXES :

In the frame S', let us consider the quantity,

$$x'^2 - w'^2 = x'^2 - c^2t'^2$$

$$= \left(\frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}\right)^2 - c^2\left(\frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}}\right)^2$$

$$= \frac{(x - Vt)^2}{1 - \frac{V^2}{c^2}} - \frac{c^2\left(t - \frac{Vx}{c^2}\right)^2}{1 - \frac{V^2}{c^2}}$$

$$= \frac{(x^2 - 2xVt + V^2t^2) - c^2\left(t^2 - \frac{2xVt}{c^2} + \frac{V^2x^2}{c^4}\right)}{1 - \frac{V^2}{c^2}}$$

$$= \frac{x^2 - 2xVt + V^2t^2 - c^2t^2 + 2xVt - \frac{V^2x^2}{c^2}}{1 - \frac{V^2}{c^2}}$$

$$= \frac{x^2\left(1 - \frac{V^2}{c^2}\right) + t^2(V^2 - c^2)}{1 - \frac{V^2}{c^2}}$$

$$= x^2 + \frac{t^2(V^2 - c^2)}{(c^2 - V^2)} . c^2$$

$$= x^2 - \frac{c^2t^2(c^2 - V^2)}{(c^2 - V^2)}$$

$$= x^2 - c^2t^2$$

$$= x^2 - w^2$$

Hence the quantity $x^2 - w^2$ remains invariant under Lorentz transformations of space and time co-ordinates.

Now, let us draw two branches of the hyperbola $x^2 - w^2 = 1$ and two branches of the hyperbola $w^2 - x^2 = 1$ as shown in Fig. 5.2. They intersect the X-axis and W-axis at the points P_1 and P_2 respectively.

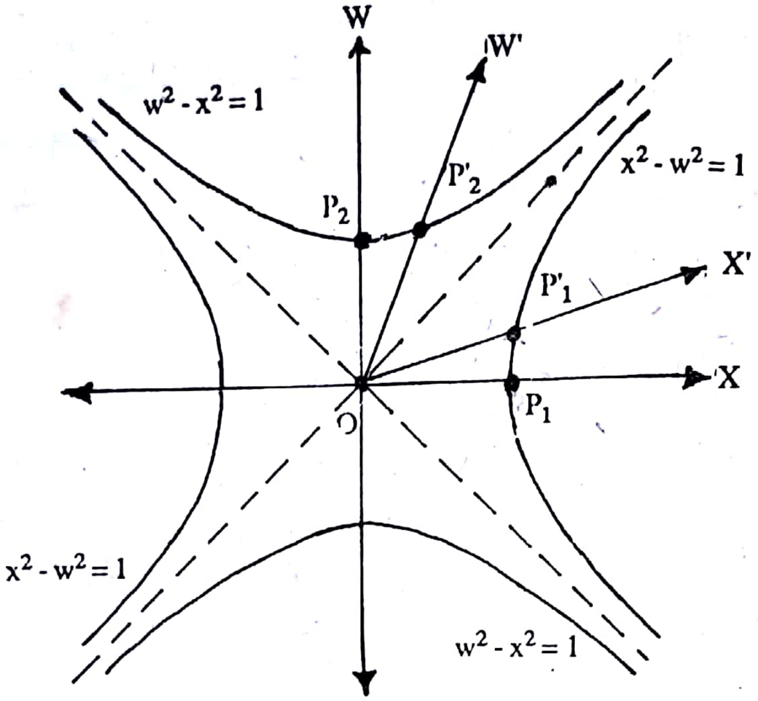


Fig. 5.2 Two branches of hyperbola $x^2 - w^2 = 1$ and of $w^2 - x^2 = 1$

For the point P_1, $w = 0$ and $x^2 - w^2 = 1$

$\therefore x^2 = 1$ and $x = 1$ i.e. $OP_1 = 1$

Using this distance OP_1 as one unit the X-axis can be calibrated as shown in Fig. 5.3.

For the point P_2, $x = 0$ and $w^2 - x^2 = 1$

$\therefore w^2 = 1$ and $w = 1$ i.e. $OP_2 = 1$

Using this distance OP_2 as one unit the W-axis can be calibrated as shown in Fig. 5.3.

Since $x^2 - w^2$ remains invariant, for the frame S' we can write,

$$x'^2 - w'^2 = x^2 - w^2 = 1 \quad \text{and}$$

$$w'^2 - x'^2 = w^2 - x^2 = 1$$

The two branches of the hyperbola intersect X'-axis and W'-axis at the point P_1' and P_2' respectively.

For the point P_1', $w' = 0$ and $x'^2 - w'^2 = 1$.

$\therefore x^2 = 1$ and $x' = 1$ i.e. $OP_1' = 1$

Using this distance OP_1' as one unit the X'-axis can be calibrated as shown in Fig. 5.3.

For the point P'_2, $x' = 0$ and $w'^2 - x'^2 = 1$

$\therefore w'^2 = 1$ and $w' = 1$ i.e. $OP'_2 = 1$

Using this distance OP'_2 as one unit the W'-axis can be calibrated as shown in Fig. 5.3.

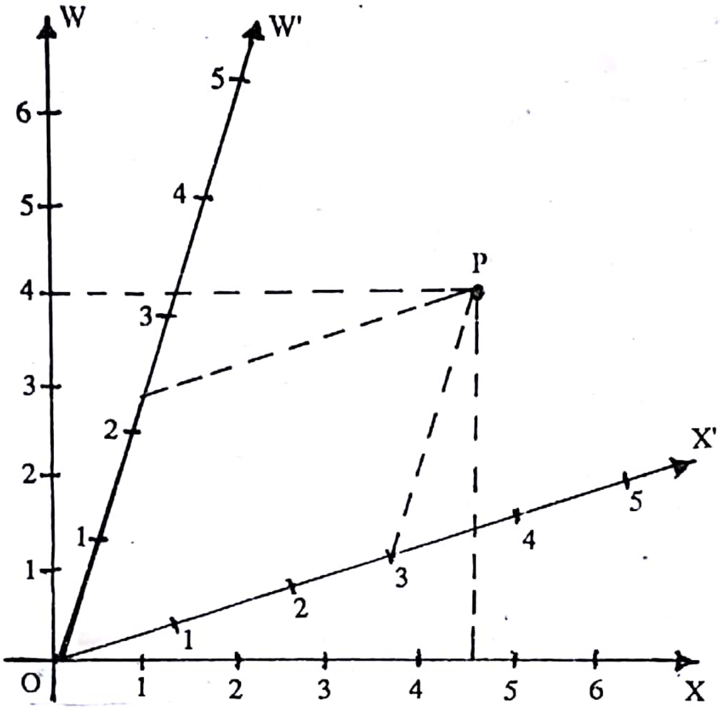


Fig. 5.3 Calibrated space and time axes of the frame S and S'

Thus using two branches of the hyperbolas we can calibrate the space and time axes of the frame S and S'. As you can see from Fig. 5.2, $OP'_1 > OP_1$ i.e. one unit of X'- axis is bigger than one unit of X-axis.

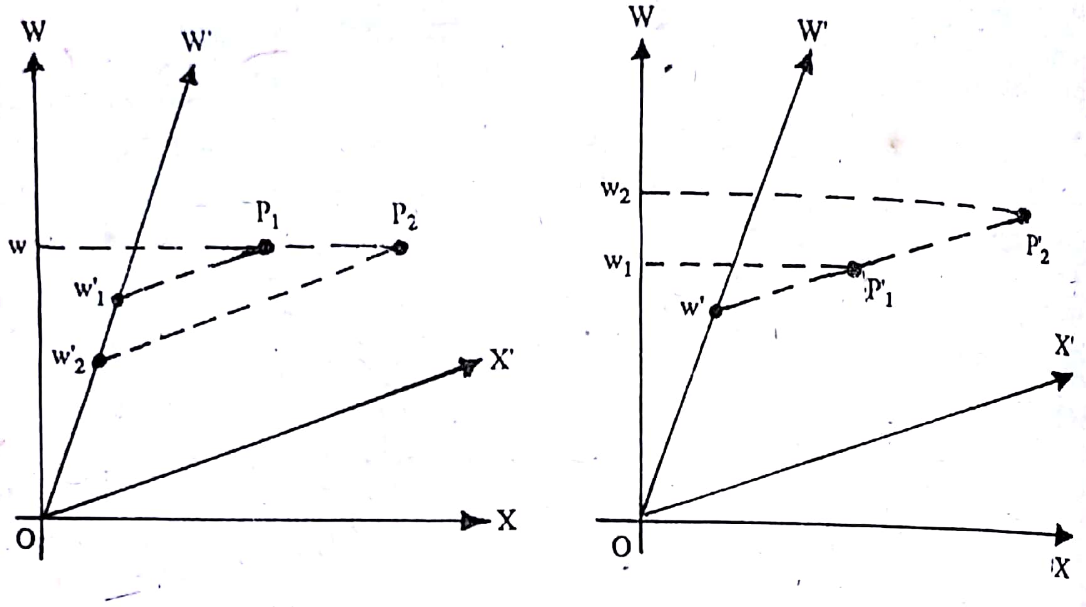
Similarly one unit of W'-axis also is bigger than one unit of W-axis.

The space and time co-ordinates of an event represented by a point P in space-time diagrams (Fig. 5.3) can be found out as follows:

Draw a line parallel to W-axis from the point P. It intersects X-axis at a point 4.5. This is the X-co-ordinate of P. Draw a line parallel to X-axis from the point P. It intersects W-axis at 4. This is the W-co-ordinate of P. Similarly you can find out X' and W'-co-ordinates of the point P for the frame S' also (See Fig. 5.3).

5.4 RELATIVITY OF SIMULTANEITY FROM SPACE-TIME DIAGRAMS :

Let us consider two events E_1 and E_2 occurring at the same instant t but at two different points relative to the frame S. The two events have the same value of $w = ct$ but their X - co-ordinates are different. They are represented by the points P_1 and P_2 respectively in the space-time diagram [Fig 5.4 (a)].



(a) (b)

Fig. 5.4 (a) Events which are simultaneous in the frame *S* are not simultaneous in the frame *S'* (b) Events which are simultaneous in the frame *S'* are not simultaneous in the frame *S*

From P_1 and P_2 draw lines parallel to X'-axis. They intersect W'-axis at w'_1 and w'_2 respectively. Hence the events are occurring at the instants $t'_1 = \frac{w'_1}{c}$ and $t'_2 = \frac{w'_2}{c}$ respectively relative to the frame S'. Since $t'_1 \neq t'_2$, the events are not simultaneous in the frame S' although they are simultaneous in the frame S.

Let us consider now the two events E_1 and E_2 occurring at the same instant t' but at two different points relative to the frame S'. The two events have the same value of $w' = ct'$ but their X' - co-ordinates are different. They are represented by the points P_1' and P_2' in the space - time diagram 5.4 (b)

From P_1' and P_2' draw lines parallel to X-axis. They intersect W-axis at w_1 and w_2 respectively. Hence the events are occurring at the instants $t_1 = \frac{w_1}{c}$ and $t_2 = \frac{w_2}{c}$ respectively relative to the frame S. Since $t_1 \neq t_2$ the events are not simultaneous in the frame S although they are simultaneous in the frame S'.

Thus, we conclude that two events occurring at different points, which are simultaneous in one frame of reference, are not simultaneous in another frame of reference in different state of motion. Hence simultaneity is a relative concept.

5.5 LORENTZ-FITZGERALD LENGTH-CONTRACTION FROM SPACE-TIME DIAGRAMS :

Let us consider an uniform rectangular bar lying along X'-axis in the frame S'. The bar is stationary relative to the frame S' and hence it is moving with velocity V along X-axis relative to the frame S.

Let x_1' and x_2' be the X-co-ordinates of the end points of the bar located along the length of the bar relative to the frame S'. Hence the proper length of the bar is given by

$$(\Delta x)_0 = x_2' - x_1'$$

The end points of the bar are represented by the world points P_1' and P_2' respectively in the space-time diagram (Fig. 5.5). Since the bar is at rest in the frame S', the world lines of the end points are parallel to W'-axis and are represented by the following equations :

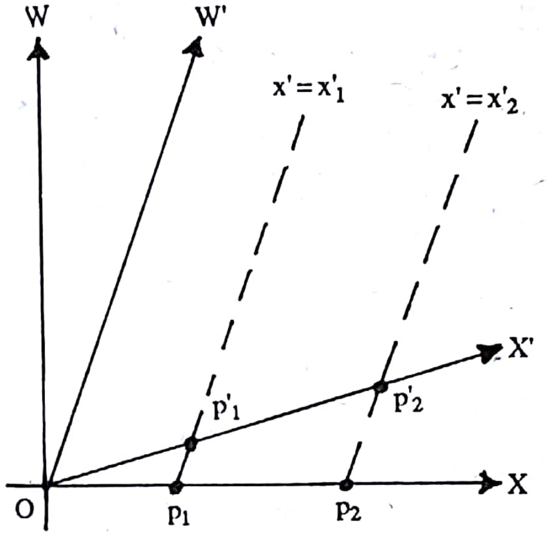


Fig. 5.5 Length -contraction from space-time diagram

$$x' = x_1' \qquad \text{and} \qquad x' = x_2'$$

They intersect X-axis at the points P_1 and P_2 respectively.

Now at the point P_1,

$$x = x_1, \quad \text{say,} \; x' = x_1' \qquad \text{and} \qquad w = 0$$

∴ The Lorentz transformation equation,

$$x' = \frac{x - \beta w}{\sqrt{1-\beta^2}} \text{ becomes}$$

$$x_1' = \frac{x_1 - \beta \times 0}{\sqrt{1-\beta^2}} = \frac{x_1}{\sqrt{1-\beta^2}}$$

At the point P_2

$$x = x_2, \quad \text{say,} \; x' = x_2' \qquad \text{and} \qquad w = 0$$

$$\therefore x_2' = \frac{x_2 - \beta \times 0}{\sqrt{1-\beta^2}} = \frac{x_2}{\sqrt{1-\beta^2}}$$

$$\therefore x_2' - x_1' = \frac{x_2}{\sqrt{1-\beta^2}} - \frac{x_1}{\sqrt{1-\beta^2}} = \frac{x_2 - x_1}{\sqrt{1-\beta^2}}$$

$$\therefore (x_2 - x_1) = (x_2' - x_1')\sqrt{1-\beta^2} = (\Delta x)_0\sqrt{1-\frac{V^2}{c^2}}$$

Now $(x_2 - x_1) = \Delta x$, the observed length of the bar relative to the frame S.

$$\therefore \Delta x = (\Delta x)_0\sqrt{1-\frac{V^2}{c^2}}$$

This is nothing but Lorentz-Fitzgerald length-contraction formula. Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1-\frac{V^2}{c^2}}$ is always less than 1.

$$\therefore \Delta x < (\Delta x)_0$$

i.e. the observed length of the moving bar is less than its proper length. Thus while moving with velocity V, the bar appears to be contracted in the direction of motion by the factor $\sqrt{1-\frac{V^2}{c^2}}$.

5.6 EINSTEIN'S TIME-DILATION FROM SPACE-TIME DIAGRAMS :

Two events E_1 and E_2 are occurring at the same point having X-co-ordinate $x' = x_1'$ relative to the frame S' but at two different instants of time t_1' and t_2' respectively. Hence the place of events is stationary relative to the frame S'. It is moving with velocity V along X-axis relative to the frame S. The proper time-interval between the two events is

$$(\Delta t)_0 = t_2' - t_1'$$

Since the place of the events is at rest in the frame S', its world line is parallel to W'-axis and is represented by the equation, $x' = x_1'$. The time co-ordinates of the events are

$$w_1' = ct_1' \text{ and } w_2' = ct_2'$$

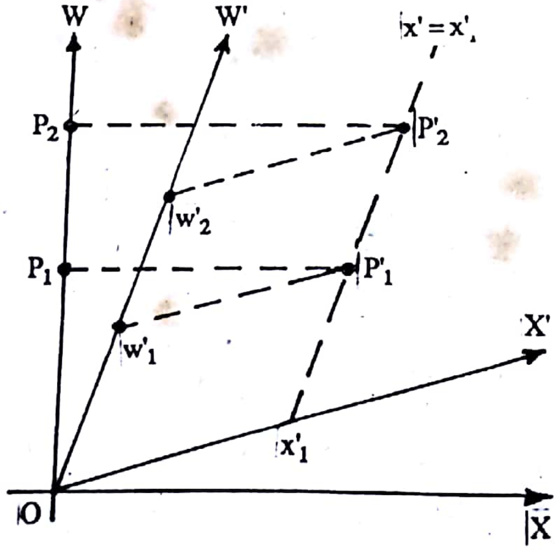


Fig. 5.6 Time-dilation from space-time diagram

The events are represented by the world points P_1' and P_2' on the world line $x' = x_1'$. From the points P_1' and P_2' draw lines parallel to X-axis. They intersect W-axis at P_1 and P_2 respectively.

Corresponding to the point P_1,

$$w = w_1, \quad \text{say}, \quad w' = w_1', \quad x' = x_1'$$

Now the Lorentz transformation equation is

$$w' = \frac{w - \beta x}{\sqrt{1 - \beta^2}}$$

Taking inverse transformations,

$$w = \frac{w' + \beta x'}{\sqrt{1 - \beta^2}}$$

$$\therefore w_1 = \frac{w_1' + \beta x_1'}{\sqrt{1 - \beta^2}}$$

Corresponding to the point P_2,

$$w = w_2, \quad \text{say}, \quad w' = w_2', \quad x' = x_1'$$

$$\therefore w_2 = \frac{w_2' + \beta x_1'}{\sqrt{1 - \beta^2}}$$

$$\therefore w_2 - w_1 = \frac{w_2' + \beta x_1'}{\sqrt{1 - \beta^2}} - \frac{w_1' + \beta x_1'}{\sqrt{1 - \beta^2}}$$

$$= \frac{w_2' + \beta x_1' - w_1' - \beta x_1'}{\sqrt{1 - \beta^2}} = \frac{w_2' - w_1'}{\sqrt{1 - \beta^2}}$$

$$\therefore ct_2 - ct_1 = \frac{ct_2' - ct_1'}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Now $t_2 - t_1 = \Delta t$, the observed time-interval between the two events relative to the frame S.

$$\therefore \Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

This is nothing but Einstein's time-dilation formula. Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1-\frac{V^2}{c^2}}$ is always less than 1.

$$\therefore \Delta t \quad > \quad (\Delta t)_0$$

i.e. if the place of events is moving the observed time-interval between two events appears to be expanded (i.e. dilated) as compared to the proper time-interval.

5.7 SPACE-TIME INTERVAL BETWEEN TWO EVENTS :

In Euclidean geometry the infinitesimal distance ds between two points is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

This distance ds remains invariant in all inertial frames of reference under Galilean transformations.

In analogy with this, the **space-time interval** ds between two events occurring infinitesimally close to each other in Minkowski world is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

Even this space-time interval ds remains invariant in all inertial frames under Lorentz transformations.

In space-time diagrams we consider only one space axis, the X-axis and ignore Y and Z-axes. Hence the above equation becomes

$$ds^2 = c^2dt^2 - dx^2$$

Moreover, $w = ct \qquad \therefore dw = c.dt$

$$ds^2 = dw^2 - dx^2$$

We have proved in § 5.3 of the present chapter that $x^2 - w^2$ remains invariant under Lorentz transformations of space and time co-ordinates. Hence $ds^2 = dw^2 - dx^2$ also remains invariant in all inertial frames.

i.e. $ds^2 = dw^2 - dx^2 = dw'^2 - dx'^2$

Due to this, whatever is true in the frame S will hold good in all other inertial frames of reference.

5.8 TIME ORDER AND SPACE SEPARATION OF EVENTS :

Let us consider one event occurring at O. Draw space and time axes with the point O as the origin.

For a flash of light starting from the origin O at $t = 0$,

$$c = \frac{x}{t} \qquad \text{i.e.} \qquad x = ct \qquad \text{or} \qquad x = w$$

Hence the world line of the flash of light emitted from O is the bisector of the angle between the space and time axes. Draw world lines of the flash of light, represented by $x = + w$ and $x = - w$ (Fig. 5.7) These lines divide the entire space-time into four regions.

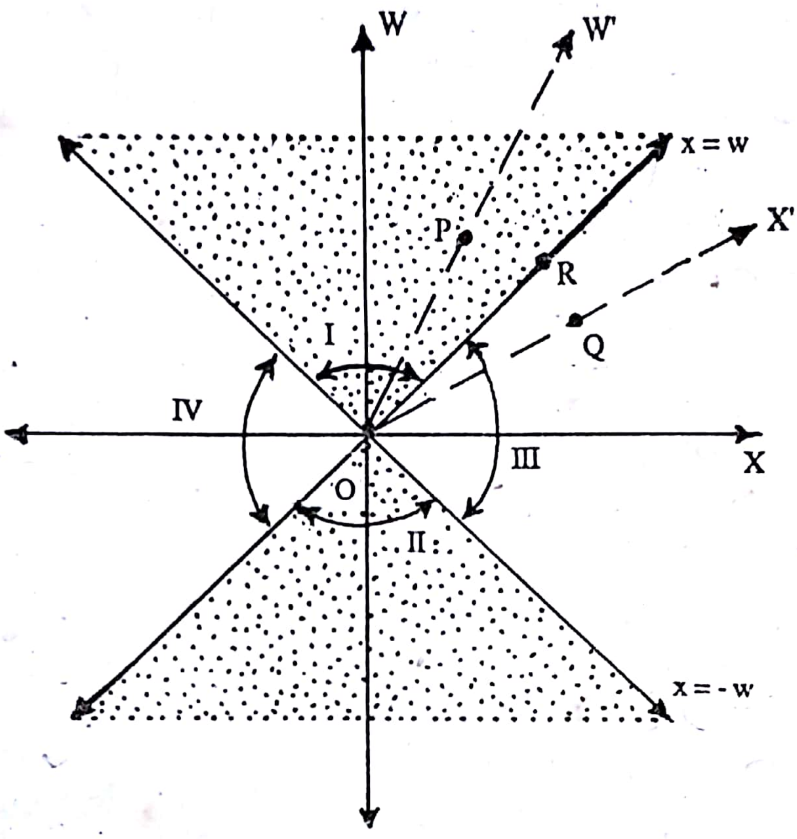


Fig. 5.7 Division of space-time into (I) absolute future, (II) absolute past and (III & IV) present.

Region I : For any event whose world point P lies in the region I, we can draw W'-axis passing through O and P. Hence there exists an inertial frame S' having X'-axis and W'-axis as space-time axes, in which the events O and P occur at the same point, $x' = 0$. They are separated **in time only.** No other event represented in this region can be simultaneous with the event O. For all such events $w > 0$ i.e. $t > 0$. Hence all events represented in this region occur after the event O. Due to this, the region I is called **absolute future** relative to O.

The space - time interval between the event O and the event P is given by

$$ds^2 = dw^2 - dx^2 = (w - 0)^2 - (x - 0)^2 = w^2 - x^2$$

For all the points P lying in the region I, $w > x$ in magnitude. Hence $w^2 > x^2$ and ds^2 is positive. Such space - time interval ds is called **time - like interval.**

Again, since $w > x$ in magnitude, $ct > x$ i.e. $c > \frac{x}{t}$ or $\frac{x}{t} < c$ i.e. any signal from O to P travels with a velocity which is less than the velocity of light c. Such signal can really exist in nature. Hence the events O and P can be causally related (i.e. they can be related with each other as a cause and effect respectively.)

Region II : Like region I, any event represented by a world point in the region II is separated from the event O in time only. For all the events represented in this region, $w < 0$ i.e. $t < 0$. Hence they occur before the event O. Due to this, the region II is called **absolute past** relative to O.

For all the points lying in this region, $w > x$ in magnitude. Hence the space-time interval is again time-like interval. The events can be causally related with O.

Thus for the events represented in regions I and II there is a definite time - order of events relative to O and they can be causally related in all inertial frames.

Regions III and IV : For any event whose world point Q lies in the region III (Fig. 5.7) we can draw x'-axis passing through O and Q. Hence there exists an inertial frame S' having x'-axis and w'-axis as space- time axes, in which the events O and Q occur at the same time, $w' = ct' = 0$. They are separated **in space only**. We can always find inertial frames in which the events represented in this region III (and similarly region IV) are simultaneous with the event O. Hence these regions are called the **Present.** For all these events there is no definite time order. The time order varies from one inertial frame to another.

The space-time interval between the event O and the event Q is given by

$$ds^2 = dw^2 - dx^2 = (w - 0)^2 - (x - 0)^2 = w^2 - x^2$$

For all the points Q lying in regions III and IV, $w < x$ in magnitude. Hence $w^2 < x^2$ and ds^2 is negative. Such space-time interval ds is called **space-like interval.**

Again, since $w < x$ in magnitude,

$$ct < x \qquad \therefore c < \frac{x}{t} \quad \text{or} \quad \frac{x}{t} > c$$

i.e. any signal from O to Q travels with a velocity which is greater the velocity of light c. According to the Special theory of Relativity such signal can not exist in nature. Hence the events O and Q can not be casually related. i.e. the events can not interact with each other as a cause and effect physically. Since the events are not causally

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related they need not have any definite time - order. Thus the Special theory of Relativity is consistent with the principle of causality.

For a world point R lying on the world line of a flash of light, $x = w$. The space time interval between the event O and the event R is given by

$$ds^2 = dw^2 - dx^2 = (w - 0)^2 - (x - 0)^2 = w^2 - x^2 = 0$$

Such space-time interval ds is called **light - like interval.**

QUESTIONS

10|5|22

1. Discuss Minkowski's four dimensional space-time continuum. [B.U. April 1988]
2. What are space-time diagrams? [B.U. Oct. 1984, April 1986, Oct. 1987, Oct. 1989, May 1990, Oct. 1990, April 1991, Oct. 1991, April 1992, May 1993, Oct. 1993]
3. Explain how space-time of Relativity can be represented geometrically. [B.U. April 1984, April 1989, Oct. 1992]
4. Using a space-time diagram illustrate how two events which are simultaneous in one inertial frame occur at different times in another inertial frame. [B.U. April 1984, April 1989, Oct. 1992]
5. Draw suitable space-time diagrams to explain the concept of relativity of simultaneity, length-contraction and time-dilation. [B.U. Oct. 1984, April 1986, Oct. 1987, Oct. 1989, May 1990, Oct. 1990, April 1991, Oct. 1991, April 1992, May 1993, Oct. 1993, May 1994]
6. What is space-time interval between two events?
7. Write a note on time order and space separation of events. [B.U. Oct. 1984, April 1986, Oct. 1989, April 1992, Oct. 1993]
8. Discuss time order and space separation of events. Bring out the meaning of "time-like" and "space-like" intervals. Drawing appropriate .diagram explain "Present", "absolute future" and "absolute past". [B.U. April 1987]
9. When is the time-interval between two events space-like? Can we establish time order between two events separated by a space-like interval? Give necessary reasoning to supplement your answer. Explain what is meant by light - like interval? [B.U. April 1984, Oct. 1992]
10. In order that two events may be causally related (i.e. related as a cause and effect) should the space-time interval between them space-like or time - lime? Explain the reason.

FOOD FOR THOUGHT

1. We have, then, the development from the three-dimensional picture of the world to the four-dimensional picture. The reader will probably not be happy with this situation, because the world still appears three-dimensional to his consciousness. How can one bring this appearance into the four - dimensional picture that Einstein requires the physicist to have?

   What appears to our consciousness is really a three-dimensional section of the four-dimensional picture. We must take a three-dimensional section to give us what appears to our consciousness at one time, at a later time we shall have a different three-dimensional section. The task of the physicist consists largely of relating events in one of these sections to events in another section referring to a later time...... Quantum theory has taught us that we have to take the process of observation into account, and observations usually require us to bring in the three-dimensional sections of the four - dimensional picture of the universe.

   - P.A.M. Dirac

^ ^ ^ ^ ^

2. In considering time as the fourth dimension more or less equivalent to the three spatial dimensions we run into one rather difficult question. When we measure length, width or height we can use in all three cases one and the same unit, say 1 inch or 1 foot. But time-duration cannot be measured either in feet or in inches

If you think little more you find a reasonable way in which a length and a duration may be compared. You often hear it said that someone lives "within twenty minutes of downtown by bus" or that some place is "only five hours away by train". Here we specify distances by giving the time necessary to cover them using a given type of transportation.

Thus if we could agree on some standard velocity we should be able to express time intervals in units of length, or vice versa. It is clear, of course, that the standard velocity to be chosen as the fundamental translation factor between space and time must be of equally fundamental and general nature. The only velocity known in Physics to possess the desired degree of generality is the velocity of light spreading through empty space. *- George Gamow*

* * * * *

3. In fact, space distances and time intervals between various events must now be considered only as projections of the basic four - dimensional separation between these events, on the space and on the time axis, so that the rotation of the four-dimensional axes - cross may result in partial transformation of distances into durations and vice versa. *- George Gamow*

* * * * *

4. Relativity theory, which has endowed classical physics with unprecedented unity and scope, has just through its elucidation of the conditions for the unambiguous use of elementary physical concepts allowed a concise formulation of the principle of causality along most general lines. *- Niels Bohr*

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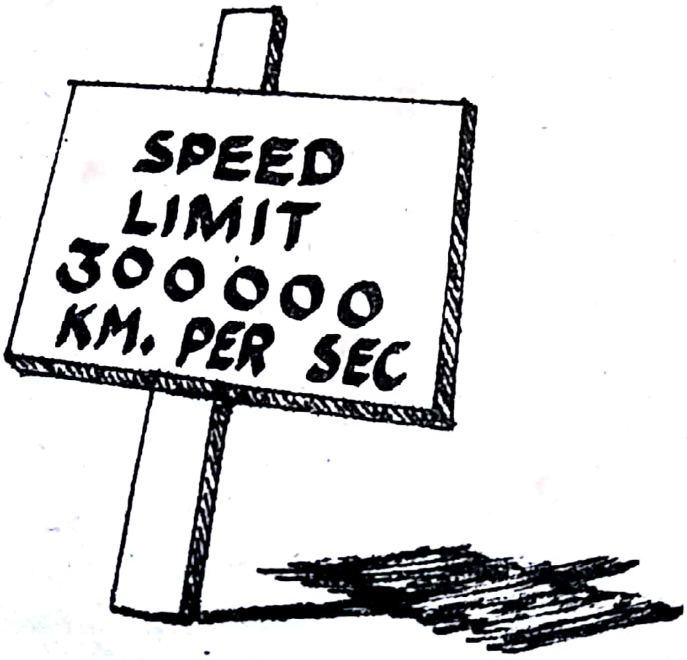


6

The Twin Paradox

In this chapter we shall study one most controversial as well as most famous topic of Relativity called twin paradox or clock paradox.

6.1 THE TWIN PARADOX :

Let us consider Einstein's time-dilation formula,

$$\Delta t = \frac{(\Delta t)_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

Since $\frac{V^2}{c^2}$ is always positive, $\sqrt{1 - \frac{V^2}{c^2}}$ is less than 1.

$$\therefore \Delta t > (\Delta t)_0$$

When the place of events is moving with velocity V, the observed time-interval Δt between two events appears to be expanded or dilated as compared to the proper time-interval between the same two events.

Due to this, one second of a moving clock is longer than one second of a stationary clock. Hence a moving clock appears to run slow as compared to a stationary clock.

In 1911, Einstein predicted the following. Let one of two twin brothers remain at home and the other go on a long round trip, travelling with very high velocity, comparable to the velocity of light. When the second twin returns home after several years, he would appear to be much younger than the twin who stayed at home.

Now, in Relativity there is neither absolute rest nor absolute motion. The motion between two frames is always relative. Hence each one of the two twins would regard himself to be stationary and the other travelling. Einstein's time-dilation phenomenon is reciprocal. For the observer in the frame S, the clock of the frame S' appears to be running slow and for the observer in the frame S' the clock of the frame S appears to be running slow. Thus for the twin at home, the travelling twin should appear to be younger than himself. For the travelling twin, the twin at home must appear to be younger than himself. Now, both of these cannot happen in reality. Logically we can say that only one can be younger than the other, both cannot be younger than each other. Such contradictory result is called twin paradox or clock paradox.

6.2 THE RESOLUTION OF TWIN PARADOX :

Such paradox arises because the travelling twin is on a round journey. When he begins his trip his frame of reference must have been accelerated heavily to attain high velocity V, comparable to the velocity of light. To return back home he must have reversed the velocity of the frame of reference from V to $-V$. This also involves high acceleration. In order to stop at home he must have retarded the frame of reference heavily. This frame is not an inertial frame at all. The twin at home is in an inertial frame of reference. Thus the two twins are not in symmetric and interchangeable situations. Hence one cannot expect the result to be reciprocal i.e. both the twins cannot be younger than each other.

The results of the Special theory of Relativity are applicable only to inertial frames of reference. If the travelling twin also would have been in an inertial frame then he would have gone away from the twin at home and would never have returned home. In that case each twin would certainly claim from his own frame of reference that the other's clock appears to be running slow as compared to his own clock and the other twin appears to be younger than himself.

Actually the travelling twin is in non-inertial frame of reference. As we shall see in the next chapter, a non-inertial frame of reference is equivalent to a gravitational field. Hence the travelling twin is subjected to a strong gravitational field. It has been found that the clocks run slow in strong gravitational field. The physical growth of the travelling twin would be slow in the strong gravitational field. This effect of gravitational field is similar to the effect of refrigeration on the growth of vegetables, fruits and meat. Hence the travelling twin would appear to be younger than the twin at home.

6.3 SPACE-TIME DIAGRAM OF TWIN PARADOX :

Initially both the twin brothers are at home represented by the world point O in the space-time diagram (Fig.6.1).

One of the twins moves along a round path and returns home after several years. He first moves to a distant point with an uniform constant velocity except at the start. This is represented by the world line OA. Then he turns back towards home. Except at the turning point and at the final point he moves with uniform constant velocity again and reaches home. This is represented by the world line AB.

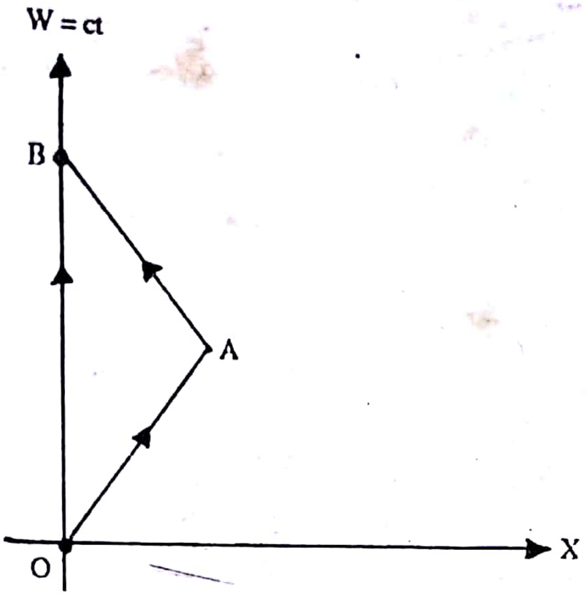


Fig 6.1 The world lines of the twins

The other twin remains at home. Since he is stationary at O, his world line is the straight line OB.

Route-Dependence of proper time :

Considering Einstein's time-dilation formula,

$$dt = \frac{dt_0}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$\therefore dt_0 = dt\sqrt{1 - \frac{V^2}{c^2}} = \sqrt{dt^2 - dt^2\frac{V^2}{c^2}}$$

But $V = \frac{dx}{dt}$ $\qquad \therefore V^2 = \frac{dx^2}{dt^2}$

$$\therefore dt_0 = \sqrt{dt^2 - \frac{dx^2}{c^2}}$$

For the travelling twin the proper time-interval of the round trip is given by

$$(\Delta t_0)_T = \int_{OAB} dt_0 = \int_{OAB} \sqrt{dt^2 - \frac{dx^2}{c^2}} \qquad ...(6.1)$$

For the twin at home, the proper time-interval of the round trip is given by

$$(\Delta t_0)_H = \int_{OB} dt_0 = \int_{OB} \sqrt{dt^2 - \frac{dx^2}{c^2}}$$

Now, over the world line OB, $dx = 0$

$$\therefore (\Delta t_0)_H = \int_{OB} \sqrt{dt^2} = \int_{OB} dt = t_B \qquad \cdots(6.2)$$

Let us compare the equations (6.1) and (6.2). Since $\frac{dx^2}{c^2}$ is always positive,

$$(\Delta t_0)_T < (\Delta t_0)_H$$

i.e.proper time of the travelling clock is less than the proper time of the clock at home. Hence the travelling twin appears to be younger than the twin at home.

Moreover the value of the proper time-interval $(\Delta t_0)_T$ of the travelling twin depends upon the path OAB along which the twin completes the round journey. This is called Route dependence of the proper time.

6.4 EXPERIMENTAL VERIFICATION :

It is not possible to devise an experiment in which we would keep one twin at home and send the other twin on a round trip in space and verify the effects directly because no spaceship can travel with a high velocity comparable to the velocity of light $c = 3 \times 10^8$ m/s.

However, experiments have been performed on radioactive nuclei using the apparatus of Mossbauer effect. It has been observed that a sample of radioactive nuclei lying on the perimeter of a revolving turntable undergo fewer decays than an identical sample kept at rest at the centre of the turntable. i.e.the rate of decay of the radioactive nuclei on the round trip is less than those at the centre. Thus the clock paradox is true. The twin on the round trip would be ageing less and hence appear younger than the twin at home.

QUESTIONS

1. What is twin paradox? Explain the resolution of twin paradox. [B.U. Oct. 1990]
2. Draw space-time diagram of twin paradox. [B.U. Oct. 1990]
3. What do you understand by "The route dependence of proper time"? [B.U. April 1985, April 1988, May 1990]
4. Write a short note on "twin paradox". [B.U.Oct.1989, April 1991, Oct. 1991, April 1992, Oct. 1992, Oct. 1993, May 1994]

FOOD FOR THOUGHT

1. We can only conclude that the special theory of relativity cannot claim an unlimited domain of validity, its results hold only so long as we are able to disregard the influences of gravitational fields on the phenomena. - *Albert Einstein*

* * *

2. The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible. - *Albert Einstein*

* * *

7

Elements of General Theory of Relativity

In the preceding chapters we have formulated the Special theory of Relativity which is valid and applicable only in inertial frames of reference. We have seen in § 1.3 of the first chapter that in reality there is no 100 % perfect inertial frame of reference in the universe. Ideal inertial frame exists only in theory. All our actual frames of reference are non-inertial. Can we formulate a theory which would be valid and applicable in all types of frames of reference, inertial as well as non-inertial? Can we build up a real relativistic Physics which would be valid in all frames of reference - whatever may be their states of motion? Then the laws of Physics would be true and identical in all frames of reference. The new theory would reduce to the Special theory of Relativity for inertial frames of reference.

Einstein took up this task and in 1916 he put forward a theory which is valid in all frames of reference. This theory is called General theory of Relativity and it is probably the most beautiful and the most perfect of all the existing theories of Physics. In the present chapter we shall discuss the elements of General theory of Relativity. It is founded on two simple postulates which are as follows.

7.1 BASIC POSTULATES OF GENERAL THEORY OF RELATIVITY :

1. Principle of General Covariance :

All the laws of Physics are invariant in all frames of reference, inertial as well as non-inertial. i.e. the equations representing the laws of Physics should have the same form in all frames of reference, whatever may be their states of motion.

2. Principle of Equivalence :

In a small region of space-time, a non-inertial frame of reference is equivalent to a gravitational field.

We shall study and discuss these two principles in detail in the following sections.

7.2 PRINCIPLE OF GENERAL COVARIANCE :

As we have seen in the second chapter, the basic philosophy of the Special theory of Relativity is that the description of physical events and the values of the physical quantities may change from one inertial frame of reference to another but all the natural laws of Physics should be invariant in all inertial frames of reference under Lorentz transformations. This philosophy led to the development of one of the most consistent theories of Physics and it yielded many epoch - making results.

The great success achieved by the Special theory of Relativity led Einstein to extend this philosophy to all frames of reference, inertial as well as non-inertial frames of reference. The principle of general covariance stresses that all laws of Physics should be formulated in such a way that the equations representing them should have the same form in all frames of reference - whatever may be their states of motion. i.e. the equations should have the covariant form with respect to any arbitrary co-ordinate transformation. This can be done by expressing the laws of Physics in the form of tensor equations. i.e. all terms in the equation should be tensors of the same rank. The reason is that a tensor equation has the same form in all co-ordinate systems and in all frames of reference.

For example, Einstein's gravitational field equation is

$$R_{ik} - \frac{1}{2} g_{ik} R = 8\pi T_{ik}$$

where T_{ik} is energy-momentum tensor of the matter producing the gravitational field, R_{ik} is Riemann curvature tensor of the corresponding space-time and g_{ik} is metric tensor of the space - time.

Each term in this equation is a tensor of rank two. When we transform the equation to some other arbitrary frame of reference S' it becomes.

$$R'_{ik} - \frac{1}{2} g'_{ik} R' = 8\pi T'_{ik}$$

Thus the tensor equation can retain its form in any frame of reference.

We know from Classical Mechanics that while constructing any equation representing a law of Physics it is always required that the terms of the equation should be all scalars or all vectors. The principle of general covariance imposes a general condition viz. the terms of the equation representing a law of Physics should be all tensors of the same rank. Then the equation will retain its form universally in all frames of reference and the law of Physics will be invariant in all frames of reference whatever may be their states of motion.

.3 PRINCIPLE OF EQUIVALENCE :

To illustrate the principle of equivalence Einstein devised one thought experiment with an elevator.

Einstein's Elevator : We know from Classical Mechanics that the acceleration due to gravity, denoted by '*g*' remains the same for bodies of different masses and different materials. This fact was first discovered by Galileo experimentally by dropping objects of different masses from the Leaning Tower of Pisa. In other words, different bodies acquire the same acceleration '*g*' (roughly equal to 9.8 m/s^2) in the gravitational field of earth irrespective of their masses.

Let us consider an elevator (i.e. a lift) at the top of a high skyscraper. Imagine that the cable of the elevator breaks and the elevator falls freely in the gravitational field of the earth.

An observer lying outside the elevator (who is looking through the window of the elevator) finds that the elevator and the observer inside it are falling down, both with the same acceleration '*g*'.

The observer inside the elevator has no information of anything happening outside his elevator. He drops an object. (see Fig. 7.1). Now, according to the outside observer this object also should fall down with the same acceleration '*g*' as that of the observer and the elevator. The inside observer finds that the object remains at the same place where he had dropped it and does not fall down to the floor of the elevator. He immediately concludes that the gravitational field is absent inside the elevator. Thus a non-inertial frame of reference (i.e. the elevator falling down with acceleration '*g*') can eliminate the gravitational field.

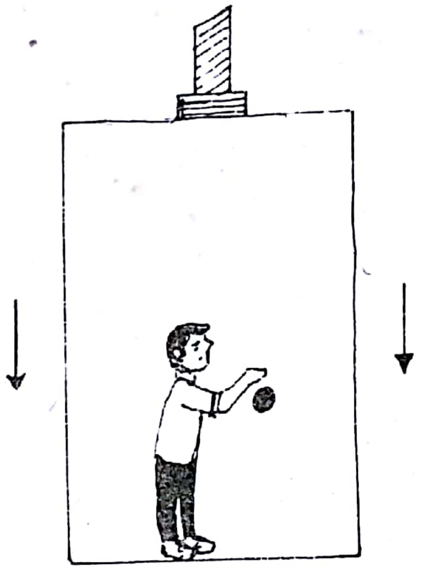


Fig. 7.1 An observer inside an elevator falling downwards

Let us now place the elevator in a region of space where the gravitational field is absent. The observer inside the elevator drops an object and finds that the object remains in the same position where he had dropped it. He concludes that the gravitational field is absent inside the elevator. Now, some person lying outside the elevator pulls up the cable of the elevator with a constant force. Hence the outside observer finds that the elevator moves up with a constant acceleration (*a*). The object obeys the law of inertia and remains in the original position but the floor of the elevator soon reaches the object with the acceleration (a). The inner observer has no knowledge of what is happening outside the elevator. He suddenly finds that the object falls down to the floor of the elevator with an acceleration (a). He concludes that, **now**, the gravitational field exists inside the elevator. Thus a non-inertial frame of reference [i.e. the elevator moving up with the acceleration (a)] can create a gravitational field.

These two conceptual experiments show that a non-inertial frame of reference can eliminate as well as create a gravitational field in a region of space-time. Hence a non-inertial frame of reference is equivalent to a gravitational field in a region of space- time.This is the substance of the Principle of equivalence.

Rotating Non-inertial Frames of Reference :

A rotating frame of reference (for example a rotating marry-go-round) is a non-inertial frame of reference. A body lying in such a frame of reference experiences centrifugal force directed away from the axis of rotation. It is given by

$$F = mr\omega^2$$

where m is the mass of the body, r is the distance of the body from the axis of rotation and ω is angular velocity of the rotating frame relative to a stationary frame.

The centrifugal acceleration experienced by the body is given by

$$a = r\omega^2$$

which does not depend upon the mass of the body. i.e. bodies of different masses located at the same distance from the axis of rotation attain the same acceleration. In this respect it is similar to the acceleration due to gravity 'g'. Hence the bodies behave as if they are lying in a gravitational field which is directed away from the axis of rotation. As we move away from the axis of rotation the centrifugal acceleration increases, or equivalently centrifugal force increases. This can be interpreted by saying that the intensity of the gravitational field present in the rotating frame of reference goes on increasing from zero as we move away from the axis of rotation of the frame. Thus even a rotating frame of reference is equivalent to a gravitational field.

A frame of reference which is moving with uniform linear acceleration is equivalent to a constant, uniform gravitational field in the corresponding region of space-time. If the acceleration of the frame of reference changes then the frame is equivalent to uniform but time - varying gravitational field.

It should be emphasized that a non-inertial frame of reference is equivalent to a gravitational field in a small region of space-time but it is not identical with the actual gravitational field. For example, at infinite distances from the bodies producing the gravitational field the actual gravitational field becomes zero. But the gravitational field to which a non-inertial frame of reference is equivalent does not become zero at infinity.

According to Principle of equivalence, all that can be done is to eliminate the gravitational field in a certain small region of space-time where the field can be considered to be uniform, by introducing a non-inertial frame of reference whose acceleration is the same as that which would be acquired by a particle lying in the above gravitational field.

Thus as soon as we try to extend our theory of Relativity from inertial frame of reference to a non-inertial frame of reference we have to deal with gravitational field. We cannot escape from the gravitational field because, according to the principle of equivalence, a non-inertial frame of reference is as good as a gravitational field in a small region of space - time. The gravitational field becomes one of the properties of a non-inertial frame of reference. Hence the General theory of Relativity automatically leads to the formulation of a new theory of gravitation.

7.4 EQUALITY OF INERTIAL AND GRAVITATIONAL MASS :

Inertial mass : A force F_i called inertial force acts on a body. The body acquires an acceleration a_i. Then by Newton's second law of motion

$$F_i = m_i a_i$$

$$\therefore m_i = \frac{F_i}{a_i}$$

The quantity m_i given by the above formula is called inertial mass of the body.

Gravitational mass : Let the same body be kept in a gravitational field where the intensity of the field is I_g. Then a force F_g, called gravitational force acts on the body which is given by

$$F_g = m_g I_g$$

$$\therefore m_g = \frac{F_g}{I_g}$$

The quantity m_g given by the above formula is called gravitational mass of the body.

Now, let the inertial force F_i be produced by the gravitational field i.e. $F_i = F_g$

$$\therefore m_i a_i = m_g I_g$$

Then a_i is the acceleration due to gravity i.e. $a_i = g$

$$\therefore m_i \cdot g = m_g \cdot I_g$$

$$\therefore g = \frac{m_g}{m_i} I_g$$

While discussing the principle of equivalence, we have seen that it has been observed experimentally that the acceleration due to gravity (g), remains the same for bodies of different masses and different materials. All different bodies acquire the same acceleration (g) in the gravitational field of the earth irrespective of their masses. i.e. g remains constant at a point in the gravitational field. Now, I_g, the intensity of the field always remains constant at a point. Hence $\frac{m_g}{m_i}$ should remain the same for all bodies of different masses. i.e. the ratio $\frac{m_g}{m_i}$ is constant and by selecting proper units we can make that constant to be 1.

$$\therefore \frac{m_g}{m_i} = 1 \qquad \text{i.e.} \quad m_g = m_i$$

Hence the gravitational mass of a body is always equal to its inertial mass. This equality of the gravitational mass and inertial mass of a body is sometimes called Principle of equivalence. The equality has been verified experimentally by Eotvos in 1890 and by Dicke in 1964 to a very high degree of accuracy.

7.5 SPACE - TIME GEOMETRY IN THE GRAVITATIONAL FIELD :

As we have seen in the second chapter (see § 2.8), in Newtonian Relativity, the space of physical phenomena is Euclidean space which is a three dimensional continuum. The distance (ds) between two points of the space lying infinitesimally close to each other is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

According to Special theory of Relativity the space and time are intimately and cohesively connected with each other and they can be converted into each other during the co-ordinate transformations form one inertial frame to the other inertial frame. The world of physical phenomena is Minkowski space which is four dimensional space - time continuum. The space - time interval (ds) between two events occurring infinitesimally close to each other is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

The Euclidean space consists of X - Y, Y - Z, and Z - X planes. Hence it is a flate space. Minkowski space-time is a pseudo - Euclidean space in the sense that ds^2 is the sum of the squares of the co-ordinate differentials. Even this space-time is flat.

The geometry of the space-time changes drastically when the gravitational field is present in the space-time or equivalently when we consider the corresponding non-inertial frame of reference. The geometry becomes completely non-Euclidean. To understand this, let us consider the following thought experiment.

There are two frames of reference S and S'. Their origins are coinciding with each other and their Z-axes also are coinciding with each other. The frame S is an inertial frame whereas the frame S' is rotating relative to S about their common Z-axis.

A circle in X - Y plane of the frame S with centre at origin can be regarded as a circle in X' - Y' plane of the frame S' also. The observer in the frame S measures the circumference L and the diameter D of the circle using a scale of unit length. He then finds the ratio of $\frac{L}{D}$ which obviously comes out as

$$\frac{L}{D} = \pi,$$

in accordance with Euclidean geometry.

Now, the observer in the rotating frame S' also measures the circumference L' and diameter D' of the circle of the frame S' in the same manner.

Observing this process of measurement from the frame S we find that when the scale of unit length is kept along the circumference it is lying in the direction of the tangential velocity V and hence its length decreases due to Lorentz-Fitzgerald length-contraction. Therefore L' comes out to be greater than L. When the scale of unit length is kept along the diameter it is lying in the direction perpendicular to the tangential velocity V and hence its length remains unchanged. Due to this, D' comes out to be equal to D. The ratio of the circumference to diameter is

$$\frac{L'}{D'} > \frac{L}{D} = \pi, \quad \text{i.e.} \quad \frac{L'}{D'} > \pi$$

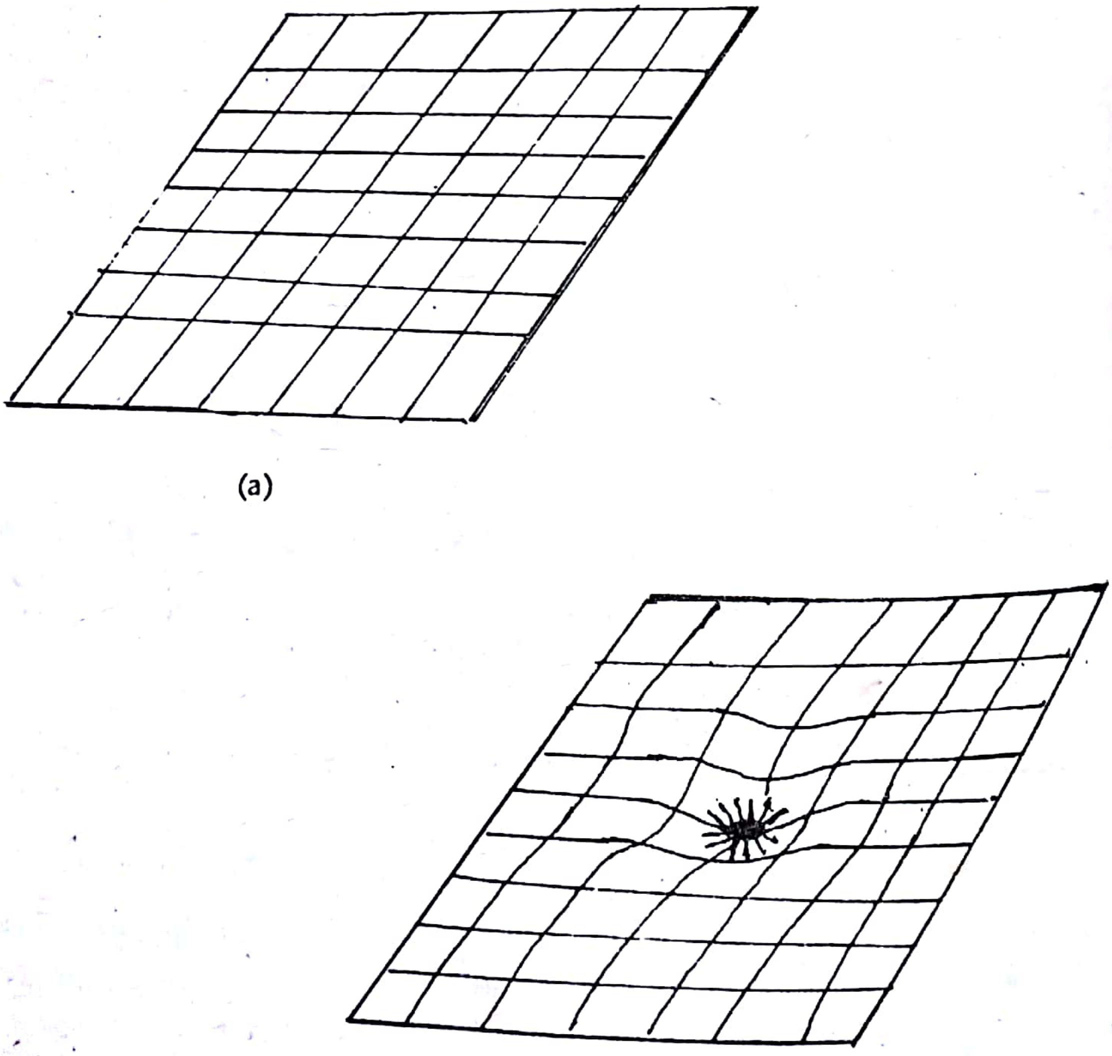
Thus the ratio is greater than π. This experiment strikingly shows that the Euclidean geometry is valid in the inertial frame of reference S but it does not hold good in the non-inertial frame S', i.e. in the equivalent gravitational field. Thus in the presence of a gravitational field the space becomes non-Euclidean.

Let us keep a clock at the centre of the circle in the frame S', and a similar clock at the circumference of the circle. Now, the clock at the centre is at rest whereas the clock at the circumference is moving with the tangential velocity *V*. Hence we observe that due to Einstein's time - dilation this clock runs slow as compared to the clock at the centre. The clock at the circumference is lying in the equivalent gravitational field directed away from the axis of rotation whereas the clock at the centre is free from the gravitational field. Thus we conclude that a **clock lying in a gravitational field appears to run slow.**

These experiments clearly establish the following :

In the presence of gravitational field the space-time geometry becomes non-Euclidean. It is not even Minkowskian.

According to Einstein's theory of gravitation, in the absence of matter there is no gravitational field and the space-time continuum is flat. [See Fig. 7.2 (a)] Here Euclidean geometry holds good.


(a)

(b)

Fig. 7.2 (a) Flat space-time continuum in the absence of matter.
(b) Curved space-time continuum in the presence of the matter (in two dimensions)

As soon as matter enters the space-time continuum, the continuum becomes curved. The presence of matter causes the space-time continuum to bend [See Fig. 7.2 (b)]. The bigger the mass the larger is the curvature of the continuum. On the curved space-time Euclidean geometry is not valid.

For example, the sum of the three angles of a triangle drawn on a plane is always equal to 180°. This simple theorem of Euclidean geometry is not valid if the triangle is drawn on the surface of a sphere as shown in Fig. 7.3. For curved space-time we shall have to devise a new geometry.

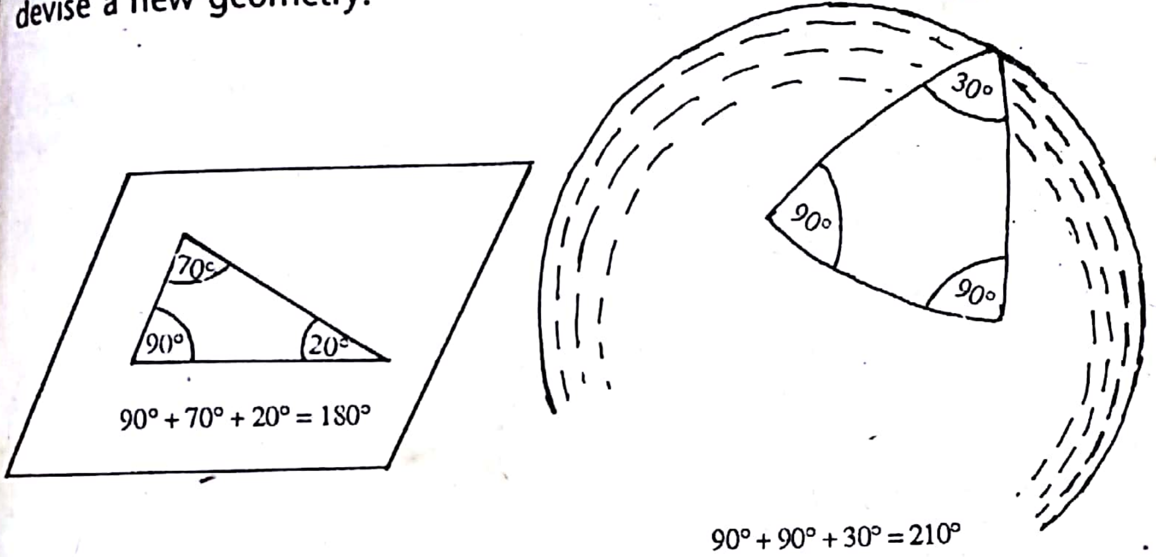


Fig. 7.3 Euclidean geometry is true on a plane surface but not on the surface of a sphere

7.6 RIEMANNIAN SPACE - TIME CONTINUUM :

In a frame of reference rotating with angular velocity ω about Z-axis, ds^2 is given by

$$ds^2 = [c^2 - \omega^2 (x^2 + y^2)]\, dt^2 - dx^2 - dy^2 - dz^2 + 2\omega y\, dx.dt - 2\omega x\, dy.dt$$

Here ds^2 is not the sum of the squares of the co-ordinate differentials. It also contains the product of the co-ordinate differentials such as $dx.dt$, $dy.dt$. Hence in a general non-inertial frame of reference ds^2 is expressed in quadratic form of general co-ordinate differentials as follows :

$$ds^2 = \sum_{ik} g_{ik} . dx_i . dx_k$$

where g_{ik} are certain functions of the co-ordinates x_i. In space-time co-ordinates, $x_1 = x$, $x_2 = y$, $x_3 = z$ and $x_0 = ct$.

Thus in a non-inertial frame of reference, or equivalently in the presence of a gravitational field, the space-time continuum is curved and not flat like Minkowskian. The co-ordinates x_i are called curvilinear co-ordinates. The quantities g_{ik} form a metric tensor of the space-time and it determines the geometric properties of the curved space-time. Such space-time is called Riemannian space-time.

For inertial frames of reference,

$$g_{00} = +1, \quad g_{11} = g_{22} = g_{33} = -1, \quad g_{ik} = 0 \quad \text{for } i \neq k$$

$$\begin{aligned}\therefore ds^2 &= g_{00}\, dx_0^2 + g_{11}\, dx_1^2 + g_{22}\, dx_2^2 + g_{33}\, dx_3^2 \\ &= (cdt)^2 - 1\,.\, dx_1^2 - 1\,.\, dx_2^2 - 1.dx_3^2 \\ &= c^2dt^2 - dx^2 - dy^2 - dz^2\end{aligned}$$

This shows that Minkowskian space - time (and Euclidean space - time also) is a special case of Riemannian space-time. Thus Riemannian space-time is general and universal space-time continuum. Since the General theory of Relativity covers all frames of reference; inertial as well as non-inertial, the space - time of the frames of reference is Riemannian space - time continuum.

7.7 EINSTEIN'S GENERAL LAW OF MOTION :

In Euclidean space, the shortest path between two points is a straight line joining the two points because the space is flat. In Riemannian space the shortest path between two points is a curve because the space itself is curved. For example, on the surface of a sphere the shortest path connecting two points is an arc of a circle passing through the two points. This shortest path is called **geodesic.**

According to Classical Mechanics, every body produces a gravitational field around itself and through this gravitational field the body exerts a gravitational force of attraction on other objects lying in the field. While developing the General theory of Relativity, Einstein discarded the idea of the gravitational field by saying that there is nothing like a gravitational field. The body under consideration merely bends the space-time continuum around itself. [See Fig. 7.2 (b)] The continuum becomes curved. The bigger the mass of the body the larger is the curvature of the space-time continuum. An object moving in this curved space-time will describe a curved path which is a geodesic on the space-time. Einstein formulated the following general law of motion which replaces Newton's laws of motion and gravitation :

Every body moves along a geodesic in the four dimensional space - time continuum provided it does not undergo collision with another body and is not acted upon by electromagnetic forces.

According to Einstein's law, it is not correct to say that the sun exerts gravitational force of attraction on the planets through the gravitational field, due to which the planets describe circular orbits around it. In reality, the sun merely curves the space - time continuum around it due to its high mass. [See Fig. 7.2 (b)] The planets move along geodesics in the space-time continuum, which are circles in the continuum. For example, an ant moving on a glass sphere in the same direction describes a circle because the space in which it is moving is spherical and not due to any centripetal force acting on it. But if the sphere is not visible one would say that since the ant is moving in a circle some centripetal force must be acting on it due to gravity. Precisely this happened in Newton's theory of gravitation. Whether a planet is moving around the sun or a satellite is moving from the earth to moon or a ray of light travels from the source of light to the observer, every object is simply moving along a geodesic of the space-time continuum whose shape and curvature is governed by the distribution of the mass in the space-time continuum.

Thus the General theory of Relativity eliminates the idea of gravity and gravitational force from Mechanics and replaces it with the geometry of the space - time continuum the curvature of which is decided by the mass of the matter present in the continuum. Thus geometry becomes a branch of Physics .

Note : We cannot carry on the discussion on the formulation of the General theory of Relativity beyond this limit because it requires advanced knowledge of Tensor algebra and Tensor calculus. For further studies, the interested students are requested to read the second half of "The Classical theory of Fields", an excellent book on the subject written by L.D. Landau and E.M. Lifshitz. At the end we shall present some important and epoch - making achievements of the General theory of Relativity.

7.8 IMPORTANT CONCLUSIONS OF GENERAL THEORY OF RELATIVITY :

One of the most fascinating features of the General theory of Relativity is that the results deduced from it are deviated only slightly from the results of Newton's theory of gravitation. For weak gravitational fields they reduce to the results of Newton's theory i.e. the old theory is the special limiting case of the new. Hence all astronomical observations which support the Newton's theory also support the General theory of Relativity. However, in strong gravitational fields the General theory of Relativity accounts for some special, extremely accurate phenomena which could not be explained by Newton's theory. Thus the General theory of Relativity does not overthrow Newton's theory of gravitation but it is rather a refinement of Newton's theory for strong gravitational fields. In cosmological phenomena where we have to deal with intense gravitational fields of the stars and galaxies the new theory yields excellent results. In what follows we shall present three important, strikingly accurate conclusions of the General theory of Relativity which have been tested by the experiments.

i) The Motion of the Perihelion of Mercury :

According to Newton's theory of gravitation a planet moves around the sun in an elliptical orbit and the position of this orbit remains permanently fixed relative to the sun.

According to Einstein's General theory of Relativity, the elliptical orbit of every planet should rotate around the sun in the course of time as shown in Fig. 7.4.

Perihelion of a planet is the point in its orbit, which is nearest to the sun. According to the General theory of Relativity, when the elliptical orbit rotates around the sun, this perihelion moves from P_1 to P_2. According to Newton's theory of gravitation, the elliptical orbit remains fixed permanently and hence the perihelion of each planet should remain stationary.

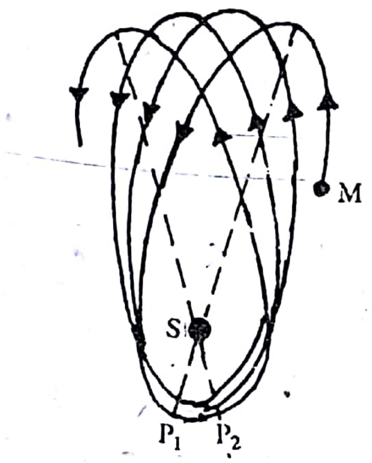


Fig. 7.4 Advance of the Perihelion of Mercury M form P_1 to P_2

Now, the planet Mercury M being closest to the sun moves in a very strong gravitational field of the sun. Hence the effects predicted by the General theory of Relativity can be appreciable and the deviations from Newton's law of gravitation can be tested.

According to the theoretical calculations of the General theory of Relativity the perihelion of Mercury should advance by 43 seconds of arc per century. For other planets the advance of perihelion is too small to be detected because they move in weak gravitational field of the sun.

Leverrier and other astronomers in 1859 had observed that the elliptical orbit of Mercury rotates very slowly and advance of the perihelion is 43 seconds of arc per century which is correct within a few seconds of arc. Thus there is an excellent agreement between the result of General theory of Relativity and observations.

ii) Deflection of Light in Gravitational Field :

As we have seen in § 7.7, according to Einstein's General theory of Relativity a body of high mass bends the space-time continuum around itself. The continuum becomes curved. The bigger the mass of the body the larger is the curvature of the space-time continuum. When a ray of light passes through this curved space-time, like any other object, it travels along a curved path which is its geodesic on the curved space-time continuum. In other words, the rays of light cannot travel along a straight line in a gravitational field. They are **deflected by the gravitational field from their rectilinear path.** This conclusion of the General theory of Relativity can be tested experimentally as follows :

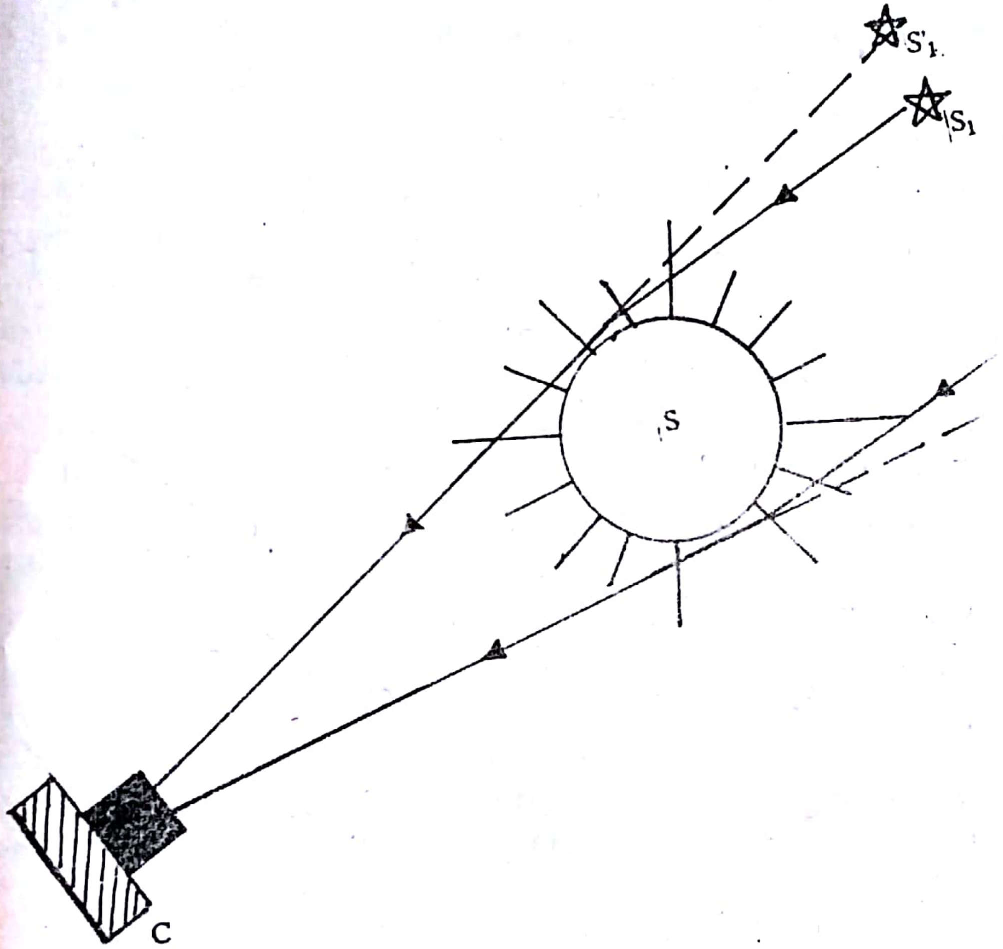


Fig. 7.5 Deflection of the rays of light coming from the stars, in the gravitational field in the neighbourhood of the sun-disc

Let us consider two stars S_1 and S_2 (Fig. 7.5) which appear to be lying in the neighbourhood of the sun-disc S, when observed from the earth. The rays emerging from the stars have to pass through the heavily curved space-time continuum (or equivalently, extremely strong gravitational field) surrounding the sun. Hence they are deflected from their rectilinear path as shown in Fig. 7.5. When observed from the earth they appear to occupy the positions S_1' and S_2' respectively. Actually, due to the brilliance of the solar disc we can not see the stars around the sun but during the total eclipse of the sun, the stars can be visible during the daytime.

In May 1919, there was total solar eclipse and British astronomers took the photographs of the apparent positions S_1' and S_2' of the stars. Thus the apparent positions of the stars were recorded from the earth. After a few months, when the sun shifted to some other position in the sky the photographs of the same stars were taken. Hence their actual positions S_1 and S_2 were located. On comparing the photographs, the deviations in the positions of the stars produced due to deflection of the rays of light in gravitational field of the sun were observed. The deviations were measured. Thus the deflection of light in the gravitational field was tested experimentally and the result of measurements confirmed the prediction of the General theory of Relativity in a thoroughly satisfactory manner.

iii) Gravitational Red - shift of the Spectral Lines :

In § 7.5 of this chapter we have come to the conclusion that a clock lying in a gravitational field appears to run slow. Hence in a strong gravitational field a clock runs more slowly than in a weak gravitational field. Now, we know that the mass of the sun is about 3×10^5 times as large as the mass of the earth. Therefore a clock placed on the sun runs more slowly than the clock placed on the earth.

Let us consider a source of sodium light on the earth. The atoms of sodium emit light which gives rise to a line spectrum in a prism-spectrometer. The spectrum consists of one line of wavelength λ.

The sodium atoms contained in the matter of the sun also emit similar light which can be received on the earth. These atoms emit the light in a relatively stronger gravitational field. According to the General theory of Relativity the period T_s of sodium light emitted from the sun is greater than the period T_e of sodium light emitted on the earth.

i.e. $T_s > T_e$

Now frequency ν is given by $\nu = \frac{1}{T}$ $\qquad \therefore T = \frac{1}{\nu}$

$$\therefore \frac{1}{\nu_s} > \frac{1}{\nu_e}$$

But $c = \nu\lambda$ gives $\nu = \frac{c}{\lambda}$

$$\therefore \frac{\lambda_s}{c} > \frac{\lambda_e}{c}$$

$$\therefore \lambda_s > \lambda_e$$

i.e. the wavelength of the sodium light coming from the sun must be greater than that from the sodium lamp on the earth. Hence in the spectrum of the light coming from the sun the spectral line due to sodium light must be slightly displaced towards the red end of the visible spectrum as compared to the line spectrum of the light from the sodium lamp on the earth. This displacement is called gravitational red-shift of the spectral line. Whatever is true for sodium is true for all other elements.

In General theory of Relativity the formula giving the red-shift of the spectral lines has been derived and such red-shift has been calculated for the sun. The experimental observations have confirmed this type of shift of the spectral lines. For the stars, the red-shift is quite large due to their high mass. Even this has been observed experimentally.

These experimental observations have, no doubt, supported the General theory of Relativity but this theory is considered as one of the greatest intellectual achievements of the present century because of its inner consistency and simplicity as well as generality of its fundamental assumptions.

QUESTIONS

1. State the basic postulates of the General theory of Relativity.
2. State and explain the Principle of general covariance.
3. State and explain the Principle of equivalence.
4. Describe Einstein's thought experiments to establish equivalence between a non - inertial frame of reference and gravitational field in a small region of space.
5. What is inertial mass and gravitational mass of a body? Prove that they are equal.
6. Write a short note on equivalence between inertial mass and gravitational mass. [B.U. Oct. 1989]
7. Explain with suitable examples how the space - time geometry becomes non - Euclidean (or non - Minkowskian) in the presence of a gravitational field.
8. Write a short note on Riemannian space - time continuum.
9. What is a geodesic?
10. State and explain Einstein's general law of motion.
11. What are the important conclusions derived from the General Rela[illegible]y? What are the experimental observations in favour of these conclusions?
12. Write short notes on the following :
    a) Motion of the perihelion of Mercury,
    b) Deflection of light in gravitational field,
    c) Gravitational red - shift of the spectral lines. [B.U. Oct. 1991, May 1994]
    d) Principle of equivalence of General Relativity. [B.U. May 1989, May 1993, Oct. 1993]
    e) Einstein's General theory of Relativity. [B.U. April 1991, April 1992, Oct. 1993, May 1994]

FOOD FOR THOUGHT

1. Can we formulate physical laws so that they are valid for all co-ordinate systems, not only those moving uniformly, but also those moving quite arbitrarily, relative to each other? If this can be done, our difficulties will be over. We shall then be able to apply the laws of nature to any co-ordinate system.

   - Albert Einstein & Leopold Infeld

* * * * *

2. Natural laws must be covariant with respect to arbitrary continuous transformations of the co-ordinates. This requirement limits the natural laws concerned incomparably more strongly than the special principle of Relativity.

   - Albert Einstein

* * * * *

3. In contrast to electric and magnetic fields, the gravitational field exhibits a most remarkable property, which is of fundamental importance for what follows. Bodies which are moving under the sole influence of a gravitational field receive an acceleration which does not in the least depend either on the material or on the physical state of the body.
- Albert Einstein

* * * * *

4. Einstein made another most important contribution to the development of our physical picture : he put forward the General Theory of Relativity, which requires us to suppose that the space of Physics is curved. Before this, physicists had always worked with a flat space, the three - dimensional flat space of Newton which was then extended to the four-dimensional flat space of special theory of Relativity. General Relativity made a really important contribution to the evolution of our physical picture by requiring us to go over to curved space. The general requirement of this theory means that all the laws of physics can be formulated in curved four - dimensional space, and that they show symmetry among the four dimensions.
- P.A.M. Dirac

* * * * *

5. Thus the concept of gravity as an independent force completely disappears from our reasoning, and is replaced by the concepts of the pure geometry of space - time in which all material objects move along the geodesics, following the curvature produced by the presence of other big masses.
- George Gamow

* * * * *

6. Since the gravitational field is determined by the configuration of masses and changes with it, the geometric structure of this space is also dependent on physical factors. Thus, according to this theory, space is no longer absolute, its structure depends on physical influences. Physical geometry is no longer an isolated self - contained science like the geometry of Euclid.
- Albert Einstein

* * * * *

7. It is true that the theory of Relativity, particularly the General theory, has played a rather modest role in the correlation of empirical facts so far, and it has contributed little to atomic physics and our understanding of quantum phenomena. It is quite possible, however, that some of the results of the General theory of Relativity, such as the general covariance of the laws of nature and their nonlinearity, may help to overcome the difficulties encountered at present in the theory of atomic and nuclear processes. Apart from this, the theory of Relativity has a special appeal because of its inner consistency and logical simplicity of its axioms.
- Albert Einstein

* * * * *

8. I know from my own painful searching, with its many blind alleys, how hard it [illegible] take a reliable step, be it ever so small, towards the understanding of that which is truly significant.
- Albert Einstein

* * * * *

* * *

EPILOGUE

We are coming to the end of our journey in Relativity. If you have properly understood Relativity you must have realized that Relativity was born and has grown up on one basic philosophy : The natural laws of Physics are invariant first in all inertial frames of reference under Lorentz transformations and then in all general frames of reference, whatever may be their states of motion, under any arbitrary co-ordinate transformations. This philosophy has taken us to what excellent heights of our intellect! It created revolution not only in our fundamental concepts of Physics such as simultaneity, space, time, mass,energy, gravity etc. but also established symmetrical connections between space and time, momentum and energy, electric and magnetic field intensities. It has established equivalence between space and time, mass and energy, electrostatic force and electrodynamic force, gravitational field and a non-inertial frame of reference. In trying to implement this philosophy we have gone from three dimensional Euclidean flat space to four dimensional Minkowskian space-time continuum and from there to a general Riemannian curved space-time continuum.

The most important aspect of all these developments and achievements is that they were first attained by the genius of a single man, Albert Einstein who says : "The simpler our picture of the external world and the more facts it embraces, the more strongly it reflects in our minds the harmony of the universe". And to visualize this harmony of the universe he created the theories of Relativity, which are considered as the greatest achievements of human intellect of all time. Robert Oppenheimer has rightly said : "Deep understanding of what it means that no signal could travel faster than light would surely have come, the formal equations were already known, but this simple brilliant understanding of Physics could well have been slow in coming, had Einstein not done it for us, so beautifully, so naturally!!".

But still this is not the end of our struggle to understand the persuits of nature. According to Einstein, "In our endeavour to understand the reality we are somewhat like a man trying to understand the mechanism of a closed watch. He sees the face and the moving hands, even hears its ticking but he has no way of opening the case. If he is ingenious he may form some picture of a mechanism which could be responsible for all the things he observes, but he may never be quite sure his picture is the only one which could explain his observations. But he certainly believes that as his knowledge increases, his picture of reality will become simpler and simpler and will explain a wider and wider range of his sensuous impressions. He may also believe in the existence of the ideal limit of knowledge and that it is approached by the human mind. He may call this ideal limit as the objective Truth". Let us pray thus :

आ नो भद्राः क्रतवो यन्तु विश्वतः ।

Let noble thoughts come to us from every side. - *Rigveda 1-89-i*

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